# VECTOR ALGEBRA

## 1.1 SCALARS AND VECTORS

Vector analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best comprehended.

**A scalar** is a quantity that has only magnitude

Quantities such as time, mass, distance, temperature, electric potential, and population are scalars.

A **vector** is a quantity that has both magnitude and direction

Vector quantities include velocity, force, displacement, and electric field intensity.

A **field** is a function that specifies a particular quantity everywhere in a region

If the quantity is scalar (or vector), the **field** is said to be a scalar (or vector) field. Examples of **scalar fields** are temperature distribution in a building, sound intensity in a theater, electric potential in a region, and refractive index of a stratified medium. The gravitational force on a body in space and the velocity of raindrops in the atmosphere are examples of **vector fields**.

## 1.2 UNIT VECTOR

Let **A** be a vector $\mathbf{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$ The *magnitude* of **A** is a scalar written as A or $|\vec{A}|$, which is given by

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

A *unit vector* $\boldsymbol{a}_A$ along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**, that is,

$$\boldsymbol{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\vec{A}}{|\vec{A}|}$$

or

$$\boldsymbol{a}_A = \frac{\mathrm{A}_{\boldsymbol{x}}\boldsymbol{a}_x + \mathrm{A}_{\boldsymbol{y}}\boldsymbol{a}_y + \mathrm{A}_{\boldsymbol{z}}\boldsymbol{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Note that $|\boldsymbol{a}_A| = 1$.

## 1.3 VECTOR ADDITION AND SUBTRACTION

Two vectors **A** and **B** can be added together to give another vector **C**; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

The vector addition is carried out component by component. Thus, if

$\vec{\mathrm{A}} = \mathrm{A}_{\boldsymbol{x}}\boldsymbol{a}_x + \mathrm{A}_{\boldsymbol{y}}\boldsymbol{a}_y + \mathrm{A}_{\boldsymbol{z}}\boldsymbol{a}_z$ and $\vec{\mathrm{B}} = \mathrm{B}_{\boldsymbol{x}}\boldsymbol{a}_x + \mathrm{B}_{\boldsymbol{y}}\boldsymbol{a}_y + \mathrm{B}_{\boldsymbol{z}}\boldsymbol{a}_z$.

$$\vec{\mathrm{A}} + \vec{\mathrm{B}} = (\mathrm{A}_{\boldsymbol{x}} + B_x)\boldsymbol{a}_x + (\mathrm{A}_{\boldsymbol{y}} + B_y)\boldsymbol{a}_y + (\mathrm{A}_{\boldsymbol{z}} + B_z)\boldsymbol{a}_z$$

Vector subtraction is similarly carried out as

$$\vec{\mathrm{A}} - \vec{\mathrm{B}} = (\mathrm{A}_{\boldsymbol{x}} - B_x)\boldsymbol{a}_x + (\mathrm{A}_{\boldsymbol{y}} - B_y)\boldsymbol{a}_y + (\mathrm{A}_{\boldsymbol{z}} - B_z)\boldsymbol{a}_z$$

## 1.3 POSITION AND DISTANCE VECTORS

The **position vector $\boldsymbol{r}_p$. (**or **radius vector)** of point $P$ is defined as the directed distance from the origin O to $P$, that is,

$$\boldsymbol{r}_p = OP = x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z$$

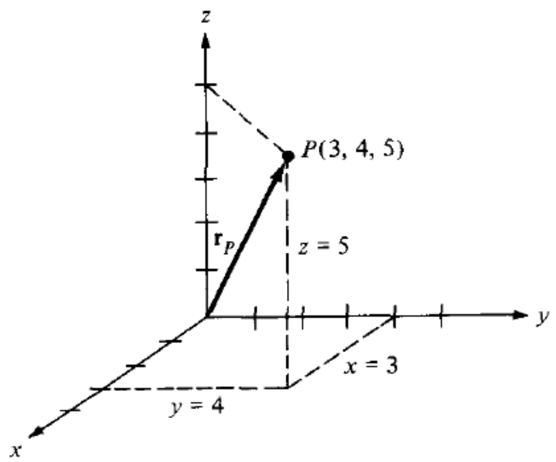



**Figure 1.1** Illustration of position vector $\boldsymbol{r}_p = 3\boldsymbol{a}_x + \boldsymbol{4}\boldsymbol{a}_y + 5\boldsymbol{a}_z$.

The **distance vector** is the displacement from one point to another

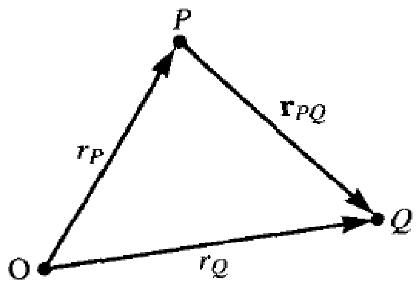



**Figure 1.2** Distance vector $\boldsymbol{r}_{pQ}$.

If two points $P$ and $Q$ are given by $(x_P, y_P, z_P)$ and $(x_Q, y_Q, z_Q)$, the **distance vector** is the displacement from $P$ to $Q$ as shown in Figure 1.2; that is,

$$\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_p$$

$$= (x_Q - x_P)\boldsymbol{a}_x + (y_Q - y_P)\boldsymbol{a}_y + (z_Q - z_P)\boldsymbol{a}_z$$

***Example:*** Find the vector between the points P (1, 4, 2) and Q (3, 1, 6)?

***Solution:*** $\overrightarrow{PQ} = (3,1,6) - (1,4,2)$

$$\overrightarrow{PQ} = (3-1)\boldsymbol{a}_x + (1-4)\boldsymbol{a}_y + (6-2)\boldsymbol{a}_z = 2\boldsymbol{a}_x - 3\boldsymbol{a}_y + 4\boldsymbol{a}_z$$

$$\overrightarrow{QP} = (1,4,2) - (3,1,6)$$

$$\overrightarrow{QP} = (1-3)\boldsymbol{a}_x + (4-1)\boldsymbol{a}_y + (2-6)\boldsymbol{a}_z = -2\boldsymbol{a}_x + 3\boldsymbol{a}_y - 4\boldsymbol{a}_z$$

***Example:*** If $\boldsymbol{A} = 10\boldsymbol{a}_x - 4\boldsymbol{a}_y + 6\boldsymbol{a}_z$ and $\boldsymbol{B} = 2\boldsymbol{a}_x + \boldsymbol{a}_y$ find:

(a) the component of A along $\boldsymbol{a}_y$, (b) the magnitude of $\boldsymbol{A}$,

(c) the magnitude of $\boldsymbol{B}$, (d) the magnitude of $\boldsymbol{3A} - \boldsymbol{B}$ ,

(e) a unit vector along $\boldsymbol{A}$.

***Solution:*** (a) The component of $A$ along $\boldsymbol{a}_y$ is $A_y = -4$ .

(b) $A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{10^2 + 4^2 + 6^2} = 12.32$

(c) $B = \sqrt{2^2 + 1^2} = 2.23$

(d) the magnitude of $\boldsymbol{3A} - \boldsymbol{B}$

$$\boldsymbol{3A} - \boldsymbol{B} = 3\left(10\boldsymbol{a}_x - 4\boldsymbol{a}_y + \boldsymbol{6a}_z\right) - (2\boldsymbol{a}_x + \boldsymbol{a}_y) = 28\boldsymbol{a}_x - 13\boldsymbol{a}_y + 18\boldsymbol{a}_z$$

$$|\boldsymbol{3A} - \boldsymbol{B}| = \sqrt{28^2 + 13^2 + 18^2} = \sqrt{1277} = 35.74$$

(e) $\boldsymbol{a}_A = \dfrac{\vec{A}}{\left|\vec{A}\right|} = \dfrac{10\boldsymbol{a_x} - 4\boldsymbol{a_y} + 6\boldsymbol{a_z}}{12.32}$

$$\boldsymbol{a}_A = 0.812\boldsymbol{a}_x - 0.32\ \boldsymbol{a}_y + 0.49\ \boldsymbol{a}_z$$

$$|\boldsymbol{a}_A| = \sqrt{0.812^2 + 0.32^2 + 0.49^2} = 1$$

***Example :*** Points *P* and *Q* are located at (0, 2, 4) and ( - 3 , 1, 5). Calculate

(a)The position vector *P*

(b) The distance vector from *P* to *Q*

(c) The distance between *P* and *Q*

(d) A vector parallel to *PQ* with magnitude of 10

***Solution:***

(a) $\vec{P} = 0\boldsymbol{a}_x + 2\boldsymbol{a}_y + 4\boldsymbol{a}_z = 2\boldsymbol{a}_y + 4\boldsymbol{a}_z$

(b) $\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (-3, 1, 5) - (0, 2, 4)$

$$= -3\boldsymbol{a}_x - \boldsymbol{a}_y + \boldsymbol{a}_z$$

(c) Since $\boldsymbol{r}_{PQ}$ is the distance vector from *P* to *Q*, the distance between *P* and *Q* is the magnitude of this vector; that is,

$$d = |\boldsymbol{r}_{PQ}| = \sqrt{9 + 1 + 1} = 3.317$$

Alternatively:

$$d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2} = \sqrt{9 + 1 + 1} = 3.317$$

(d) Let the required vector be $\vec{A}$, then

$$\boldsymbol{a}_A = \frac{\vec{A}}{|\vec{A}|} \;\rightarrow\; \vec{A} = |\vec{A}|\boldsymbol{a}_A$$

where $|\vec{A}|$ = 10 is the magnitude of $\vec{A}$. Since $\vec{A}$ is parallel to *PQ*, it must have the same unit vector as $\boldsymbol{r}_{PQ}$ or $\boldsymbol{r}_{QP}$. Hence,

$$\boldsymbol{a}_A = \boldsymbol{a}_{PQ} = \frac{\boldsymbol{r}_{PQ}}{|\boldsymbol{r}_{PQ}|} = \frac{-3\boldsymbol{a}_x - \boldsymbol{a}_y + \boldsymbol{a}_z}{3.317}$$

$$\vec{A} = \frac{10(-3\boldsymbol{a}_x - \boldsymbol{a}_y + \boldsymbol{a}_z)}{3.317} = (-9.045\boldsymbol{a}_x - 3.015\boldsymbol{a}_y + 3.015\boldsymbol{a}_z)$$

## 1.6 VECTOR MULTIPLICATION

When two vectors **A** and **B** are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\mathbf{A} \cdot \mathbf{B}$
2. Vector (or cross) product: $\mathbf{A} \times \mathbf{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

or

4. Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

### A. The Dot Product

The dot product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \cdot \vec{B}$, is defined geometrically as the dot product of the magnitude of $\vec{B}$ and the projection of $\vec{A}$ onto $\vec{B}$ (or vice versa)

Thus:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta_{AB}$$

where $\boldsymbol{\theta_{AB}}$ is the smaller angle between **A** and **B**, the result of $\mathbf{A} \cdot \mathbf{B}$ is called either *scalar product* since it is scalar, or *dot product* due to sign.

If $\vec{A} = A_{\boldsymbol{x}}\boldsymbol{a}_x + A_{\boldsymbol{y}}\boldsymbol{a}_y + A_{\boldsymbol{z}}\boldsymbol{a}_z$ and $\vec{B} = B_{\boldsymbol{x}}\boldsymbol{a}_x + B_{\boldsymbol{y}}\boldsymbol{a}_y + B_{\boldsymbol{z}}\boldsymbol{a}_z$ then

$$\vec{A} \cdot \vec{B} = A_{\boldsymbol{x}}B_x + A_{\boldsymbol{y}}B_y + A_{\boldsymbol{z}}B_z$$

Not that

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_y = \boldsymbol{a}_y \cdot \boldsymbol{a}_z = \boldsymbol{a}_z \cdot \boldsymbol{a}_x = 0 \qquad (\theta = 90, \quad \cos\theta, \quad \cos 90 = 0)$$

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_x = \boldsymbol{a}_y \cdot \boldsymbol{a}_y = \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1 \qquad (\theta = 0, \quad \cos\theta\,, \quad \cos 0 = 1)$$

***Example:*** The three vertices of a triangle are located at $A(6,-1,2)$, $B(-2,3,-4)$ and $C$(-3, 1, 5). Find: (a) $\vec{R}_{AB}$ ; (b) $\vec{R}_{AC}$ ; (c) the angle $\theta_{BAC}$ at vertex $A$ ?

***Solution:***

(a) $\vec{R}_{AB} = (-2,3,-4) - (6,-1,2)$

$$= (-2-6)\boldsymbol{a}_x + \big(3-(-1)\big)\boldsymbol{a}_y + (-4-2\big)\boldsymbol{a}_z$$

$$= -8\boldsymbol{a}_x + 4\boldsymbol{a}_y - 6\boldsymbol{a}_z$$

(b) $\vec{R}_{AC} = (-3,1,5) - (6,-1,2)$

$$= (-3-6)\boldsymbol{a}_x + \big(1-(-1)\big)\boldsymbol{a}_y + (5-2\big)\boldsymbol{a}_z$$

$$= -9\boldsymbol{a}_x + 2\boldsymbol{a}_y + 3\boldsymbol{a}_z$$

(c) $\left|\vec{R}_{AB}\right| = \sqrt{8^2+4^2+6^2} = 10.77$

$$\left|\vec{R}_{AC}\right| = \sqrt{9^2+2+3^2} = 9.69$$

$$\vec{R}_{AB}\cdot\vec{R}_{AC} = A_{\boldsymbol{x}}B_x + A_{\boldsymbol{y}}B_y + A_{\boldsymbol{z}}B_z$$

$$= -8(-9) + 4(2) - 6(3) = 62$$

$$\vec{R}_{AB}\cdot\vec{R}_{AC} = \left|\vec{R}_{AB}\right|\left|\vec{R}_{AC}\right| \cos\theta_{BAC}$$

$$62 = 10.77(9.69)\cos\theta_{BAC}$$

$$\cos\theta_{BAC} = \frac{62}{104.36}$$

$$\theta_{BAC} = \cos^{-1} 0.594$$

$$\therefore \theta_{BAC} = 53.55^{\circ}$$

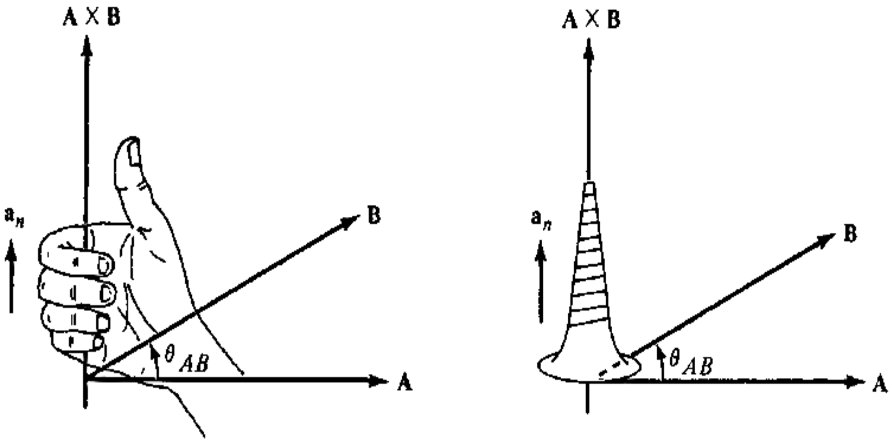
## B. Cross Product

The cross product of two vectors **A** and **B**, written as $\vec{A} \times \vec{B}$, is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta_{AB}\, \boldsymbol{a}_n$$

Where is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$ **.** The direction of is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Fig. 1.3(a). Alternatively, the direction of $\boldsymbol{a}_n$ is taken as that of the advance of a right-handed screw as **A** is turned into **B** as shown in Fig. 1.3(b).

The




(a) (b)

**Figure 1.3:** Direction of $\vec{A} \times \vec{B}$ and $\boldsymbol{a}_n$ using (a) right hand rule (b) a right-handed screw.

The vector multiplication of equation above is called *cross product* due to the cross sign; it is also called *vector product* since the result is a vector.

If $\vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$ and $\vec{B} = B_x\boldsymbol{a}_x + B_y\boldsymbol{a}_y + B_z\boldsymbol{a}_z$ then

$$\vec{A} \times \vec{B} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= (A_yB_z - A_zB_y)\boldsymbol{a}_x - (A_xB_z - A_zB_x)\boldsymbol{a}_y + (A_xB_y - A_yB_x)\boldsymbol{a}_z$$

Note that

$$\boldsymbol{a}_x \times \boldsymbol{a}_x = \boldsymbol{a}_y \times \boldsymbol{a}_y = \boldsymbol{a}_z \times \boldsymbol{a}_z = 0 \qquad \theta = 0, \quad \sin\theta = \sin 0 = 0$$

$$\boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$$

$$\boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y \quad \text{while} \quad \boldsymbol{a}_y \times \boldsymbol{a}_x = -\,\boldsymbol{a}_z$$

***Example:*** If three points $A(6,-1,2)$, $B(-2,3,-4)$ and $C(-3,1,5)$.

Find $\vec{R}_{AB} \times \vec{R}_{AC}$

***Solution:***

$$\begin{aligned}\vec{R}_{AB} &= (-2,3,-4) - (6,-1,2)\\ &= (-2-6)\boldsymbol{a}_x + \big(3-(-1)\big)\boldsymbol{a}_y + (-4-2)\boldsymbol{a}_z\\ &= -8\boldsymbol{a}_x + 4\boldsymbol{a}_y - 6\boldsymbol{a}_z\end{aligned}$$

$$\begin{aligned}\vec{R}_{AC} &= (-3,1,5) - (6,-1,2)\\ &= (-3-6)\boldsymbol{a}_x + \big(1-(-1)\big)\boldsymbol{a}_y + (5-2)\boldsymbol{a}_z\\ &= -9\boldsymbol{a}_x + 2\boldsymbol{a}_y + 3\boldsymbol{a}_z\end{aligned}$$

$$\left|\vec{R}_{AB}\right| = \sqrt{8^2 + 4^2 + 6^2} = 10.77$$

$$\left|\vec{R}_{AC}\right| = \sqrt{9^2 + 2^2 + 3^2} = 9.69$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \mathrm{R}_{AB_x} & \mathrm{R}_{AB_y} & \mathrm{R}_{AB_z} \\ \mathrm{R}_{AC_x} & \mathrm{R}_{AC_y} & \mathrm{R}_{AC_z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ -8 & 4 & -6 \\ -9 & 2 & 3 \end{bmatrix}$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = [4*3 - (-6*2)]\boldsymbol{a}_x - [(-8*3) - (-6*-9)]\boldsymbol{a}_y + (-8*2 - 4(-9))\boldsymbol{a}_z$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = 24\boldsymbol{a}_x + 78\boldsymbol{a}_y + 20\boldsymbol{a}_z$$

$$\left|\vec{R}_{AB} \times \vec{R}_{AC}\right| = \sqrt{24^2 + 78^2 + 20^2} = \left|\vec{R}_{AB}\right|\left|\vec{R}_{AC}\right| \sin\theta_{BAC}$$

$$\theta_{BAC} = \sin^{-1}\frac{\left|\vec{R}_{AB}\times\vec{R}_{AC}\right|}{\left|\vec{R}_{AB}\right|\left|\vec{R}_{AC}\right|} = \sin^{-1}\frac{\sqrt{24^2+78^2+20^2}}{10.77*9.69} \qquad \therefore\ \theta_{BAC} = 53.6^{\circ}$$

***Example:*** Given vectors $\mathbf{A} = 3\boldsymbol{a}_x + 4\boldsymbol{a}_y + \boldsymbol{a}_z$ and $\mathbf{B} = 2\boldsymbol{a}_y - 5\boldsymbol{a}_z$, find the angle between A and B**.**

***Solution:***

The angle $\theta_{AB}$ can be found by using either dot product or cross product.

$$\vec{\mathrm{A}} \cdot \vec{\mathrm{B}} = |\vec{\mathrm{A}}||\vec{\mathrm{B}}| \cos \theta_{AB}$$

$$\vec{\mathrm{A}} \cdot \vec{\mathrm{B}} = (3, 4, 1) \cdot (0, 2, -5)$$

$$= 0 + 8 - 5 = 3$$

$$|\vec{\mathrm{A}}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$

$$|\vec{\mathrm{B}}| = \sqrt{0^2 + 2^2 + 5^2} = \sqrt{29}$$

$$\cos \theta_{AB} = \frac{\vec{\mathrm{A}} \cdot \vec{\mathrm{B}}}{|\vec{\mathrm{A}}||\vec{\mathrm{B}}|} = \frac{3}{\sqrt{(26)(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1} 0.1092 = 83.73°$$

Alternatively:

$$\vec{\mathrm{A}} \times \vec{\mathrm{B}} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{bmatrix}$$

$$= (-20 - 2)\boldsymbol{a}_x - (-15 - 0)\boldsymbol{a}_y + (6 - 0)\boldsymbol{a}_z$$

$$= -22\boldsymbol{a}_x + 15\boldsymbol{a}_y + 6\boldsymbol{a}_z$$

$$|\vec{\mathrm{A}} \times \vec{\mathrm{B}}| = \sqrt{(-22)^2 + 15^2 + 6^2} = \sqrt{745}$$

$$\sin \theta_{AB} = \frac{\vec{\mathrm{A}} \times \vec{\mathrm{B}}}{|\vec{A}||\vec{B}|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$

$$\theta_{AB} = \sin^{-1} 0.994 = 83.73°$$

# COORDINATE SYSTEMS AND TRANSFORMATION

## 2.1 Introduction

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or nonorthogonal.

An **orthogonal system is** one in which the coordinates are mutually perpendicular

In this chapter, we shall restrict ourselves to the three best-known coordinate systems: the Cartesian, the circular cylindrical, and the spherical.

## 2.2 The Cartesian Coordinate System $(x, y, z)$

In the Cartesian coordinate system we set up three coordinate axes mutually at right angles to each other, and call them the $x$, $y$, and $z$-axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the $x$-axis into the $y$-axis would cause a right-handed screw to progress in the direction of the $z$-axis. Figure 1.3 shows a right-handed Cartesian coordinate system.

A point is located by giving its $x$, $y$, and $z$ coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the $x$, $y$, and $z$-axes.

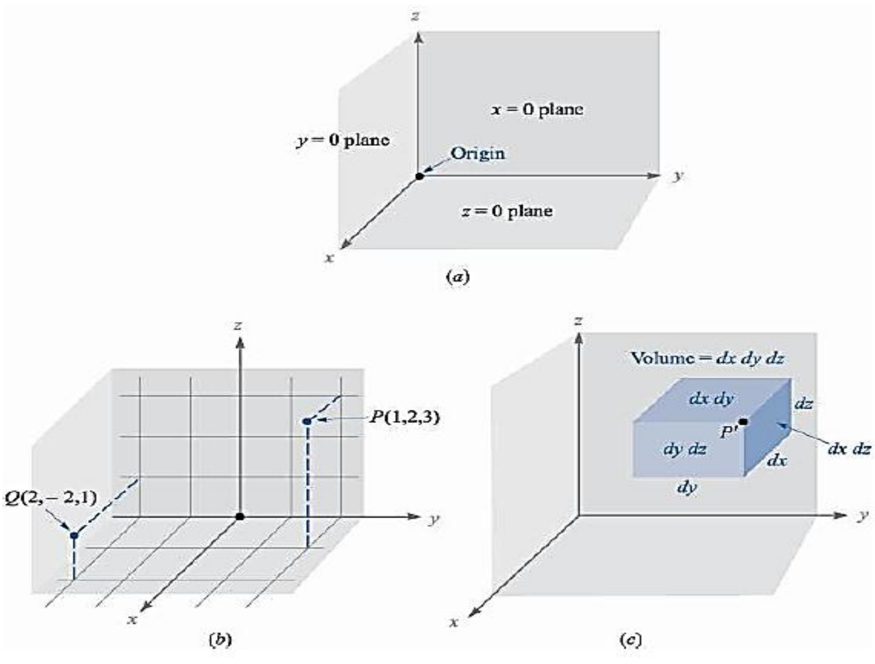



**Figure 1.3** (a) A right-handed Cartesian coordinate system. (b) The location of points P(1, 2, 3) and Q(2, -2, 1). (c) The differential volume element in Cartesian coordinates

Also shown in Figure 1.3(c) are differential element in length, area, and volume. Notes from the figure that in Cartesian coordinate:

1. Differential displacement is given by
   $d\vec{L} = dx\boldsymbol{a}_x + dy\boldsymbol{a}_y + dz\boldsymbol{a}_z$

2. Differential normal area is given by
   $d\vec{S} = dydz\ \boldsymbol{a}_x$

   $d\vec{S} = dxdz\ \boldsymbol{a}_y$

   $d\vec{S} = dxdy\ \boldsymbol{a}_z$

3. Differential volume is given by
   $dV = dx\ dy\ dz$

The ranges of the coordinate variables *x*, *y*, and *z* are

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

A vector $\vec{\mathrm{A}}$ in Cartesian coordinates can be written as shown in Figure 1.4

$$\vec{\mathrm{A}} = \mathrm{A}_x\boldsymbol{a}_x + \mathrm{A}_y\boldsymbol{a}_y + \mathrm{A}_z\boldsymbol{a}_z$$

where $\boldsymbol{a}_x$, $\boldsymbol{a}_y$, and $\boldsymbol{a}_z$ are unit vectors along the *x*, *y*, and *z*-directions.

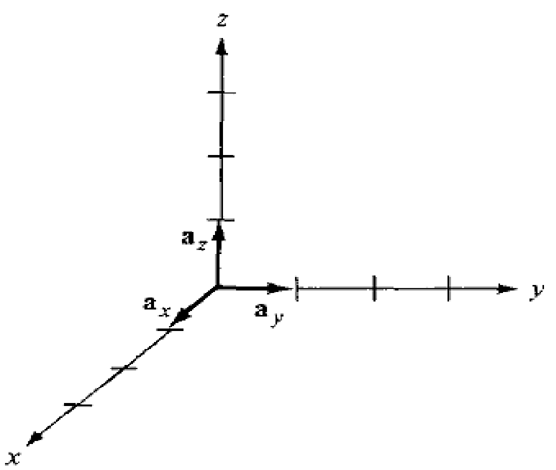



**Figure 1.4** Unit vectors $\boldsymbol{a}_x$, $\boldsymbol{a}_y$, and $\boldsymbol{a}_z$

## 2.3 The Cylindrical Coordinate System $(\rho, \emptyset, z)$

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry. A vector $\vec{A}$ in cylindrical coordinates can be written as:

$$\vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_z$$

Where $\boldsymbol{a}_\rho, \boldsymbol{a}_\emptyset,$ and $\boldsymbol{a}_z$, are unit vectors in the $\rho$,$\emptyset$, and $z$-directions as illustrated in Figure 1.5.

The magnitude of $\vec{A}$ is:

$$|\vec{A}| = \sqrt{A_\rho^2 + A_\emptyset^2 + A_z^2}$$

A point $P$ in cylindrical coordinates is represented as $(\rho, \emptyset, z)$ and is as shown in Figure 1.5. Observe Figure 1.5 closely and note how we define each space variable: $\rho$ is the radius of the cylinder passing through $P$ or the radial distance from the z-axis; $\emptyset$ is (called the *azimuthal* angle) measured from the *x*-axis in the *xy*-plane; and z is the same as in the Cartesian system.

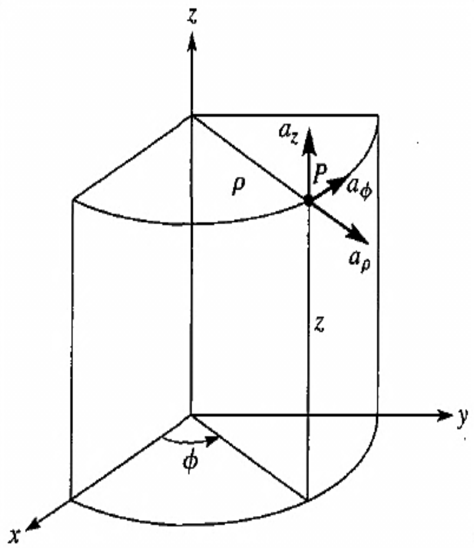



**Figure 1.5** Point *P* and unit vectors in the cylindrical coordinate system

Notice that the unit vectors $\boldsymbol{a}_\rho, \boldsymbol{a}_\emptyset$, and $\boldsymbol{a}_z$ are mutually perpendicular since our coordinate system is orthogonal; $\boldsymbol{a}_\rho$ points in the direction of increasing $\rho$, $\boldsymbol{a}_\emptyset$ in the direction of increasing $\emptyset$, and $z$ in the positive $z$-direction. Thus

$$\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\rho = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$$

$$\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_z = \boldsymbol{a}_z \cdot \boldsymbol{a}_\rho = 0$$

$$\boldsymbol{a}_\rho \times \boldsymbol{a}_\emptyset = \boldsymbol{a}_z$$

$$\boldsymbol{a}_\emptyset \times \boldsymbol{a}_z = \boldsymbol{a}_\rho$$

$$\boldsymbol{a}_z \times \boldsymbol{a}_\rho = \boldsymbol{a}_\emptyset$$

Note Also from Figure 1.5 that in cylindrical coordinate, differential element can be found:

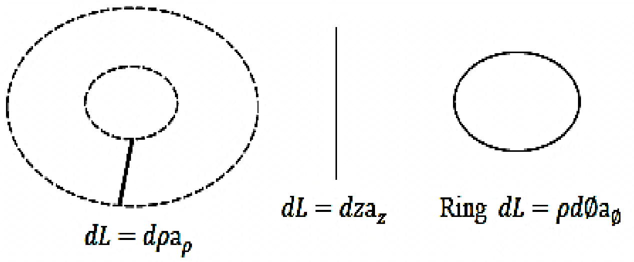
a. Differential displacement is given by:

$$d\vec{L} = d\rho\, \boldsymbol{a}_\rho$$

$$d\vec{L} = \rho\, d\emptyset\, \boldsymbol{a}_\emptyset$$

$$d\vec{L} = dz\, \boldsymbol{a}_z$$

or $d\vec{L} = d\rho\, \boldsymbol{a}_\rho + \rho\, d\emptyset\, \boldsymbol{a}_\emptyset + dz\, \boldsymbol{a}_z$

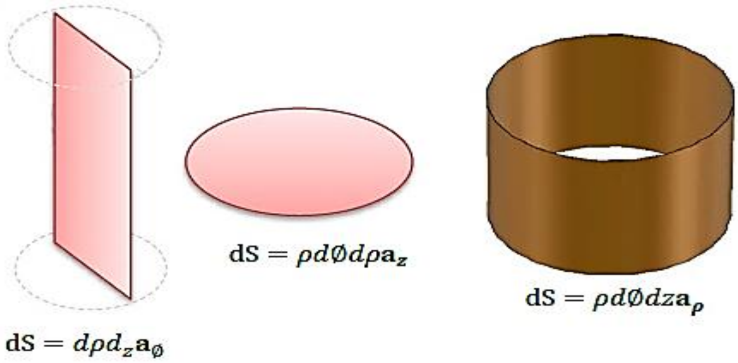



b. Differential normal area is given by:

$$d\vec{S} = \rho\, d\emptyset\, dz\, \boldsymbol{a}_\rho$$

$$d\vec{S} = d\rho\, dz\, \boldsymbol{a}_\emptyset$$

$$d\vec{S} = \rho\, d\rho\, d\emptyset\, \boldsymbol{a}_z$$




c. Differential volume is given by:

$$dV = \rho\, d\rho\, d\emptyset\, dz$$

d. The distance between two points in cylindrical coordinate $P_1(\rho_1, \emptyset_1, z_1)$ and $P_2(\rho_2, \emptyset_2, z_2)$ is given by

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 - 2\cos(\emptyset_2 - \emptyset_1) + (z_2 - z_1)^2}$$

- ***Cartesian** to **Cylindrical** Coordinate Transformation*

The relationships between the variables (*x*, *y*, *z*) of the Cartesian coordinate system and those of the cylindrical system $(\rho, \emptyset, z)$ are easily obtained as

$$\rho = \sqrt{x^2 + y^2} \quad , \quad \emptyset = \tan^{-1}\frac{y}{x}, \quad z = z$$

In matrix form, we have transformation of vector $\vec{A}$

From Cartesian coordinate $\vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$

To cylindrical coordinate $\vec{A} = A_\rho\boldsymbol{a}_\rho + A_\emptyset\boldsymbol{a}_\emptyset + A_z\boldsymbol{a}_z$ as

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- ***Cylindrical** to **Cartesian** Coordinate Transformation*

The relationships between the variables $(\rho, \emptyset, z)$ of the cylindrical coordinate system and those of the Cartesian system (*x*, *y*, *z*) are easily obtained as

$$x = \rho\ \cos\emptyset\,, \quad y = \rho\ \sin\emptyset\,, \quad z = z$$

In matrix form, we have transformation of vector $\vec{A}$

From cylindrical coordinate $\vec{A} = A_\rho\boldsymbol{a}_\rho + A_\emptyset\boldsymbol{a}_\emptyset + A_z\boldsymbol{a}_z$

To Cartesian coordinate $\vec{A} = A_x\boldsymbol{a}_x + A_y\boldsymbol{a}_y + A_z\boldsymbol{a}_z$ as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix}$$

***Example:*** Given point $P(-2, 6, 3)$ and vector $\vec{A} = y\boldsymbol{a}_x + (x+z)\boldsymbol{a}_y$ , express $P$ and $\vec{A}$ in cylindrical coordinate. Evaluate $\vec{A}$ at $P$ in Cartesian and cylindrical system?

***Solution:*** *The vector $\vec{A}$ in Cartesian coordinate at P is:*

$$\vec{A} = 6\boldsymbol{a}_x + (-2+3)\boldsymbol{a}_y = 6\boldsymbol{a}_x + \boldsymbol{a}_y$$

$$|\vec{A}| = \sqrt{6^2 + 1^2} = 6.08$$

*The point P in cylindrical coordinate is:*

$$\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + 6^2} = 6.324$$

$$\emptyset = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = z = 3$$

P(6.324 , 108.430°, 3)

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

But $x = \rho\ cos\,\emptyset,\ y = \rho\ sin\,\emptyset,\ z = z$ and substituting these yields

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\ \sin\emptyset \\ \rho\ \cos\emptyset + z \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \rho\ sin\,\emptyset\cos\emptyset + \rho\ \cos\emptyset\sin\emptyset + z\sin\emptyset \\ -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\,cos\,\emptyset \\ 0 \end{bmatrix}$$

$$A_\rho = \rho\ sin\,\emptyset\cos\emptyset + \rho\ \cos\emptyset\sin\emptyset + z\sin\emptyset$$

$$A_\emptyset = -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\,cos\,\emptyset$$

$$A_z = 0$$

$\vec{A} = A_\rho \boldsymbol{a}_\rho + A_\emptyset \boldsymbol{a}_\emptyset$

*$\vec{A}$ at point P is:*

$A_\rho = 6.324 \sin 108.43 \cos 108.43 + 6.324 \cos 108.43 \sin 108.43 + 3 \sin 108.43 = -0.948$

$A_\emptyset = -6.324 \sin^2 108.43 + 6.324 \cos^2 108.43 + 3\,cos\, 108.43 = -6.008$

$\vec{A} = -0.948\boldsymbol{a}_\rho - 6.08\boldsymbol{a}_\emptyset$

$|\vec{A}| = \sqrt{(0.948)^2 + (6.08)^2} = 6.08$

## 2.4 The Spherical Coordinate System $(r, \theta, \emptyset)$

The Spherical coordinate system is most appropriate when dealing with problems having spherical symmetry. A vector $\vec{A}$ in spherical coordinates can be written as:

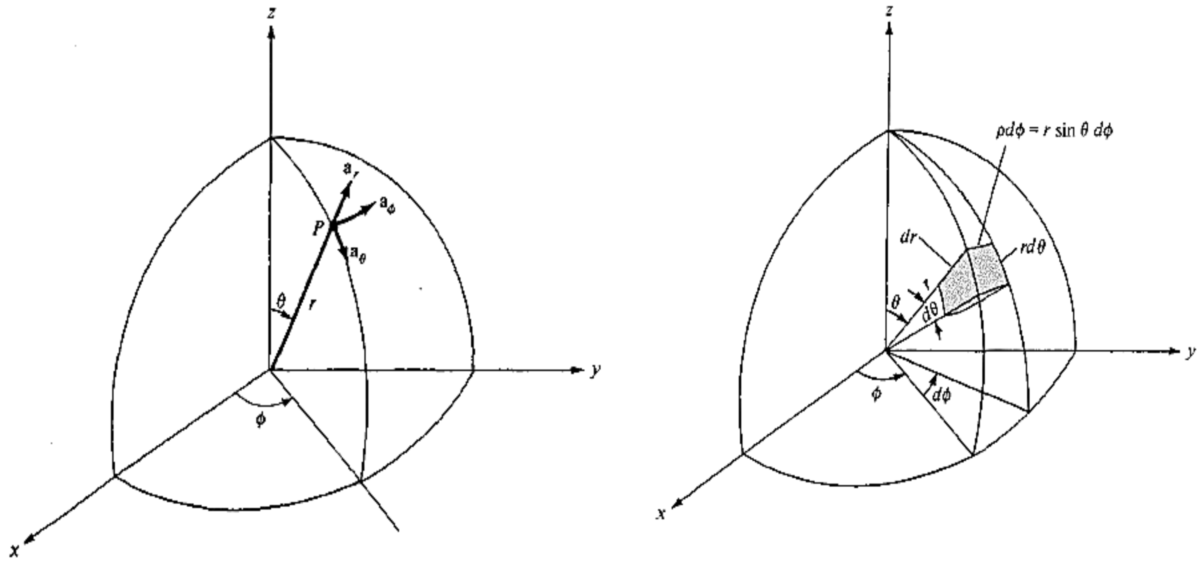
$$\vec{A} = A_r \boldsymbol{a}_r + +A_\theta \boldsymbol{a}_\theta + A_\emptyset \boldsymbol{a}_\emptyset$$

Where $\boldsymbol{a}_r, \boldsymbol{a}_\theta,$ and $\boldsymbol{a}_\emptyset$**,** are unit vectors in the $r, \theta,$ and $\emptyset$-directions

The magnitude of $\vec{A}$ is:

$$|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\emptyset^2}$$

A point $P$ in spherical coordinates is represented as $(r, \theta, \emptyset)$ and is illustrate in Figure 1.6 (a). From this Figure, we notice that $r$ is defined as the distance from the origin to the point P or the radius of a sphere centred at the origin and passing through P; $\theta$ is the angle between the z-axis and the position vector of P; and $\emptyset$ is measured from the x-axis




**(a)** **(b)**

**Figure 1.6** Spherical coordinate system **(a)** Point *P* and unit vectors **(b)** Differential elements

Notice that the unit vectors $\boldsymbol{a}_r, \boldsymbol{a}_\theta$, and $\boldsymbol{a}_\emptyset$ **,** and are mutually perpendicular since our coordinate system is orthogonal; $\boldsymbol{a}_r$ points in the direction of increasing $r$, $\boldsymbol{a}_\theta$ in the direction of increasing $\theta$ **,** and $\boldsymbol{a}_\emptyset$ in the direction of increasing $\emptyset$. Thus

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_r = \boldsymbol{a}_\theta \cdot \boldsymbol{a}_\theta = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_\emptyset = 1$$

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_\theta = \boldsymbol{a}_\theta \cdot \boldsymbol{a}_\emptyset = \boldsymbol{a}_\emptyset \cdot \boldsymbol{a}_r = 0$$

$$\boldsymbol{a}_r \times \boldsymbol{a}_\theta = \boldsymbol{a}_\emptyset$$

$$\boldsymbol{a}_\theta \times \boldsymbol{a}_\emptyset = \boldsymbol{a}_r$$

$$\boldsymbol{a}_\emptyset \times \boldsymbol{a}_r = \boldsymbol{a}_\theta$$

From Figure 1.6 (b), we note that in spherical coordinate, differential element can be found:

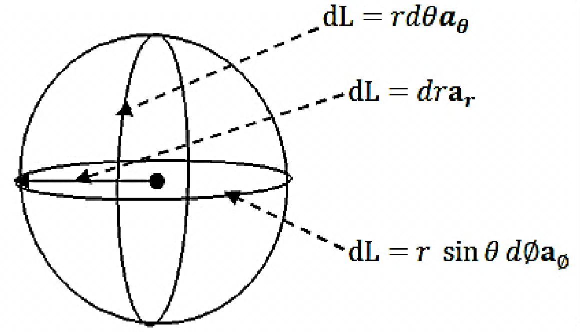
•Differential displacement is given by:

$d\vec{L} = dr\, \boldsymbol{a}_r$

$d\vec{L} = r\, d\theta\, \boldsymbol{a}_\theta$

$d\vec{L} = r \sin\theta\, d\emptyset\, \boldsymbol{a}_\emptyset$

Or $d\vec{L} = dr\, \boldsymbol{a}_r + r\, d\theta\, \boldsymbol{a}_\theta + r \sin\theta\, d\emptyset\, \boldsymbol{a}_\emptyset$

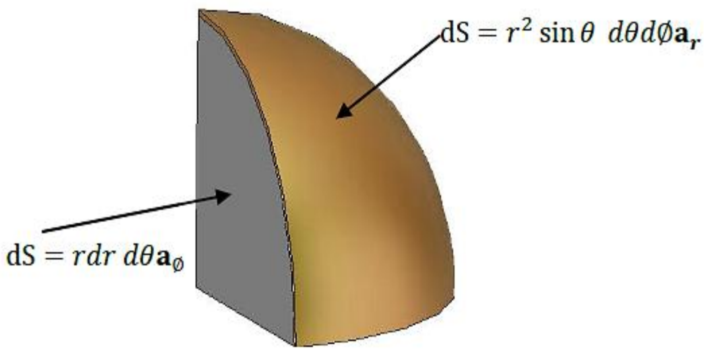



•Differential normal area is given by:

$d\vec{S} = r^2 \sin\theta\, d\theta\, d\emptyset\, \boldsymbol{a}_r$

$d\vec{S} = r \sin\theta\, dr\, d\emptyset\, \boldsymbol{a}_\theta$

$d\vec{S} = r\, dr\, d\theta\, \boldsymbol{a}_\emptyset$




•Differential volume is given by:

$dV = r^2 \sin\theta\, dr\, d\theta\, d\emptyset$

The distance between two points in spherical coordinate $P_1(r_1,\ \theta_1, \emptyset_1)$ and $P_2(r_2, \theta_2, \emptyset_2)$is given

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\emptyset_2 - \emptyset_1)}$$

- ***Cartesian** to **Spherical** Coordinate Transformation*

The relationships between the variables $(x, y, z)$ of the Cartesian coordinate system and those of the Spherical system $(r\ \theta, \emptyset)$ are easily obtained as

$$r = \sqrt{x^2 + y^2 + z^2} \qquad , \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \emptyset = \tan^{-1}\frac{y}{x}$$

In matrix form, we have transformation of vector $\vec{A}$

From Cartesian coordinate $\vec{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$

To Spherical coordinate $\vec{A} = A_r \boldsymbol{a}_r + A_\theta \boldsymbol{a}_\theta + A_\emptyset \boldsymbol{a}_\emptyset$ as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Figure 1.7 shows the relation between space variables

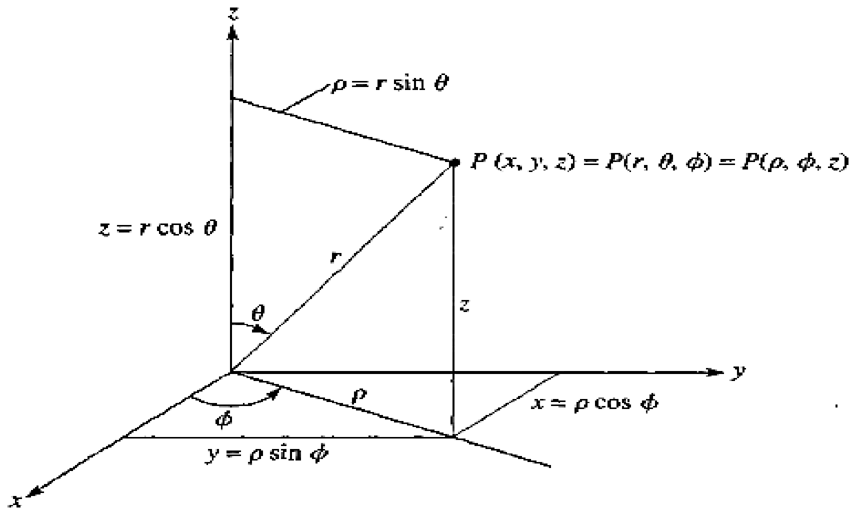



**Figure 1.7**: The relation between space variables $(x, y, z)$, $(r, \theta, \emptyset)$ and $(r, \emptyset, z)$

- ***Spherical*** *to* ***Cartesian*** *Coordinate Transformation*

The relationships between the variables ($r$, $\theta$, $\emptyset$) of the spherical coordinate system and those of the Cartesian system ($x$, $y$, $z$) are easily obtained as

$$x = r \sin\theta \cos\emptyset, \qquad y = r \sin\theta \sin\emptyset, \qquad z = r\cos\theta$$

In matrix form, we have transformation of vector $\vec{A}$

From Spherical coordinate $\vec{A} = A_r \boldsymbol{a}_r + A_\theta \boldsymbol{a}_\theta + A_\emptyset \boldsymbol{a}_\emptyset$

To Cartesian coordinate $\vec{A} = A_{\boldsymbol{x}} \boldsymbol{a}_x + A_{\boldsymbol{y}} \boldsymbol{a}_y + A_{\boldsymbol{z}} \boldsymbol{a}_z$ as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\sin\emptyset \\ \sin\theta\sin\emptyset & \cos\theta\sin\emptyset & \cos\emptyset \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix}$$

# Home Work

1. Given vectors $\mathbf{A} = \boldsymbol{a}_x + 3\boldsymbol{a}_z$ and $\mathbf{B} = 5\boldsymbol{a}_x + 2\boldsymbol{a}_y - 6\boldsymbol{a}_z$ , determine

   a) $|\mathbf{A} + \mathbf{B}|$.
   b) $\mathbf{5A - B.}$
   c) The component of $\mathbf{A}$ along $\boldsymbol{a}_y$.
   d) A unit vector parallel to $3\mathbf{A} + \mathbf{B}$.
   e) The angle $\theta_{AB}$ between the two vectors.

   **Answer:** a) 7, b) (0, - 2 , 21), c) 0, d) (0.9117, 0.2279, 0.3419) e) $\theta_{AB} = 120.6°$ .

2. Given the three points in Cartesian coordinate system as $A(3, -2, 1)$, $B(-3, -3, 5)$, $C(2, 6, -4)$. Find

   a) The vector from $A$ to $C$.
   b) The unit vector from $B$ to $A$.
   c) The distance from $B$ to $C$.

   **Answer:** a)$-\boldsymbol{a}_x + 8\boldsymbol{a}_y - 5\boldsymbol{a}_z$, b)$0.8241\boldsymbol{a}_x + 0.1373\boldsymbol{a}_y - 0.5494\boldsymbol{a}_z$, c)13.6747.

3. Transfer the vector $\vec{A} = 10\ \boldsymbol{a}_x$ to spherical coordinate at point $P(x = -3, y = 2, z = 4)$

   $$\vec{A} = -5.5702\boldsymbol{a}_r - 6.18\ \boldsymbol{a}_\theta - 5.547\boldsymbol{a}_\emptyset$$

4. Give the Cartesian coordinates of $\vec{H} = 20\ \boldsymbol{a}_\rho - 10\ \boldsymbol{a}_\emptyset + 3\ \boldsymbol{a}_z$ at point $P(x = 5, y = 2, z = -1)$

   $$\vec{H} = 22.282\ \boldsymbol{a}_x - 1.856\ \boldsymbol{a}_y + 3\ \boldsymbol{a}_z$$

# ELECTROSTATICS FIELD

## 3.1 Introduction

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields:

1. Coulomb's law
2. Gauss's law

**COULOMB'S LAW AND FIELD INTENSITY**

## 3.2 Coulomb's Law (قانون كولوم)

**Coulomb** stated that “The force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them”.

**قانون كولوم " : القوة بين جسمين صغيرين جدا يفصمهما في الفراغ أو الفضاء الحر مسافة كبيرة بالنسبة لمقاييسها تتناسب طرديا مع الشحنة على كل منهما وتتناسب عكسيا مع مربع المسافة بينهما."**

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

Where:

$F$: Force in newton (N),

$Q1$ and $Q2$ are the positive or negative quantities of charge in Coulomb(C)

$R$: is the separation in meters (m)

$\varepsilon_0$: is called the *permittivity of free space* and has the magnitude, measured in farads per meter (F/m)

$$\varepsilon_0 = 8.854 \times 10^{-12} \cong \frac{10^{-9}}{36\pi}\frac{\mathrm{F}}{\mathrm{m}}$$

or

$$k = \frac{1}{4\pi\varepsilon_0} \cong 9 \times 10^9 \ \mathrm{m/F}$$

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in mks units as $1.602 \times 10^{-19}$ C; hence a negative charge of one coulomb represents about $6 \times 10^{18}$ electrons.

If point charges $Q_1$ and $Q_2$ are located at points having position vector $\mathbf{r}_1$ and $\mathbf{r}_2$, then the vector force $\mathbf{F}_{12}$ on $Q_2$ duo to $Q_1$, shown in Figure 3.1, is given by

$$\boldsymbol{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R|^2}\mathbf{a}_{R_{12}}$$

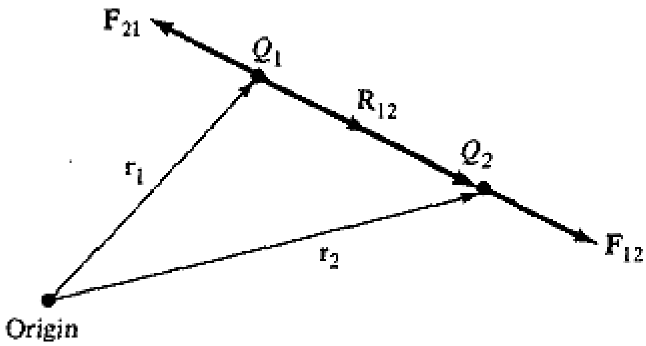



**Figure 3.1** Coulomb vector force on point charges $Q_1$ and $Q_2$

where

$$\vec{\mathrm{R}}_{12} = \vec{\mathrm{r}}_2 - \vec{\mathrm{r}}_1$$

$$\mathrm{R} = \left|\vec{\mathrm{R}}_{12}\right|$$

$$\mathbf{a}_{R_{12}} = \frac{\vec{\mathrm{R}}_{12}}{|\vec{\mathrm{R}}_{12}|}$$

$$\therefore \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R|^3}\vec{R}_{12}$$

or

$$\vec{F}_{12} = \frac{Q_1 Q_2\,(\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_1|^3}$$

As shown in Figure 3.1, the force $\vec{F}_{21}$ on $Q_1$ due to $Q_2$ is given by

$$\vec{F}_{21} = -\vec{F}_{12}$$

Like charges (charges of the same sign) repel each other while unlike charges attract. This is illustrated in Figure 3.2.


**Figure 3.2** (a), (b) Like charges repel. (c) Unlike charges attract.

From Figure 3.1 $Q_1$ located at $(x_1, y_1, z_1)$ and $Q_2$ at $(x_2, y_2, z_2)$, then

$$\vec{\mathrm{R}}_{12} = (x_2 - x_1)\boldsymbol{a}_x + (y_2 - y_1)\boldsymbol{a}_y + (z_2 - z_1)\boldsymbol{a}_z$$

$$\left|\vec{\mathrm{R}}_{12}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\mathbf{a}_{R_{12}} = \frac{\vec{\mathrm{R}}_{12}}{\left|\vec{\mathrm{R}}_{12}\right|}$$

$$\mathbf{a}_{R_{12}} = \frac{(x_2 - x_1)^2\boldsymbol{a}_x + (y_2 - y_1)^2\boldsymbol{a}_y +, (z_2 - z_1)^2\boldsymbol{a}_z}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

since

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R|^2}\mathbf{a}_{R_{12}}$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]} \frac{(x_2 - x_1)^2\boldsymbol{a}_x + (y_2 - y_1)^2\boldsymbol{a}_y + (z_2 - z_1)^2\boldsymbol{a}_z}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\mathbf{F}_2 = \frac{Q_1 Q_2 (x_2 - x_1)^2\boldsymbol{a}_x + (y_2 - y_1)^2\boldsymbol{a}_y + (z_2 - z_1)^2\boldsymbol{a}_z}{4\pi\varepsilon_0[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}$$

If we have more than two point charges, we can use *the principle of superposition* to determine the force on a particular charge. The principle states that if there are $N$ charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$ the resultant force **F** on a charge $Q$ located at point $\boldsymbol{r}$ is the vector sum of the forces exerted on $Q$ by each of the charges $Q_1, Q_2, \ldots, Q_N$ Hence:

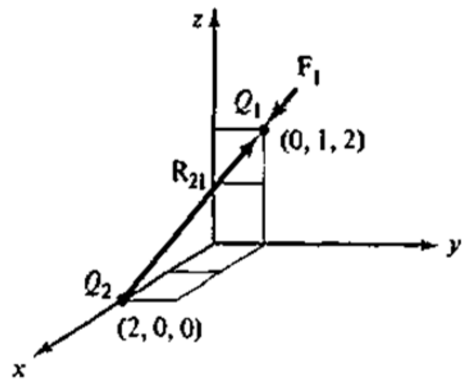
$$\vec{\mathrm{F}} = \frac{QQ_1\,(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2\,(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3} + \cdots + \frac{QQ_2\,(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3}$$

or

$$\vec{\mathrm{F}} = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k\,(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$$

***Example:*** Find the force on $Q_1$ (20µC) duo to charge $Q_2$ (-300µC), where $Q_1$ located at (0, 1, 2) and $Q_2$ at (2, 0, 0)?

***Solution:***




$$\begin{aligned}\vec{\mathrm{R}}_{21} &= (0,1,2) - (2,0,0)\\ &= (0-2)\boldsymbol{a}_x + (1-0)\boldsymbol{a}_y + (2-0)\boldsymbol{a}_z\\ &= -2\boldsymbol{a}_x + \boldsymbol{a}_y + 2\boldsymbol{a}_z\end{aligned}$$

$$\left|\vec{\mathrm{R}}_{21}\right| = \sqrt{(-2)^2 + (1)^2 + (2)^2} = 3$$

$$\mathbf{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 {R_{21}}^2}\,\mathbf{a}_{R_{21}}$$

$$\vec{\mathrm{F}}_1 = \frac{20 * 10^{-6} * (-300 \times 10^{-6})}{4\pi *\ 8.854\ * 10^{-12}\ * 3^2}\left[\frac{-2\boldsymbol{a}_x + \boldsymbol{a}_y + 2\boldsymbol{a}_z}{3}\right]$$

$$\vec{\mathrm{F}}_1 = 4\boldsymbol{a}_x - 2\boldsymbol{a}_y - 4\boldsymbol{a}_z$$

$$\left|\vec{\mathrm{F}}_1\right| = \sqrt{(4)^2 + (-2)^2 + (-4)^2} = 6\mathrm{N}$$

## 3.3 The Electric Field Intensity (E) (شدة المجال الكهربائي)

If we now consider one charge fixed in position, say $Q_1$, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force field. Call this second charge a test charge $Q_t$. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0 |R|^2} \mathbf{a}_{R_{1t}}$$

Writing this force as a force per unit charge gives

$$\frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\varepsilon_0 |R|^2} \mathbf{a}_{R_{1t}}$$

The quantity on the right side of the equation above is a function only of $Q_2$ and the directed line segment from $Q_2$ to the position of the test charge. This describes a vector field and is called the ***electric field intensity***.

Using a capital letter E for electric field intensity, we have finally

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}$$

$$\vec{\mathrm{E}} = \frac{Q_1}{4\pi\varepsilon_0 |R|^2} \mathbf{a}_{R_{1t}}$$

Where $\vec{\mathrm{E}}$ is electric field intensity measured in newtons/coulomb (N/C) or volts/meter (V/m).

The electric field intensity at point $r$ due to a point charge located at $r'$ is readily obtained from eqs.

$$\vec{\mathrm{E}} = \frac{Q(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|^3}$$

***Example:*** Find the electric field intensity (E) at (0, 2, 3) due to a point charge Q (0.4μC) located at (2, 0, 4)?

***Solution:***

$$\vec{R} = (0,2,3) - (2,0,4)$$
$$= (0-2)\boldsymbol{a}_x + (2-0)\boldsymbol{a}_y + (3-4)\boldsymbol{a}_z$$
$$= -2\boldsymbol{a}_x + \boldsymbol{2a}_y - \boldsymbol{a}_z$$

$$|\vec{R}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{R_{1t}}$$

$$\vec{E} = \frac{0.4\times10^{-6}}{4\pi\times8.854\times10^{-12}\times3^2} * \frac{-2\boldsymbol{a}_x + 2\boldsymbol{a}_y - \boldsymbol{a}_z}{3}$$

$$\vec{E} = -266.4\boldsymbol{a}_x + 266.4\boldsymbol{a}_y - 133.2\boldsymbol{a}_z$$

$$|\vec{E}| = \sqrt{(266.4)^2 + (266.4)^2 + (133.2)^2} = 399.6\ V/m$$

## 3. 4 <u>Field of N Point Charge</u>

Since the coulomb forces are linear, the electric field intensity due to $N$ point charges, $Q_1$ at $r_1$ , $Q_2$ at $r_2$,and $Q_N$ at $r_n$ is the sum of the forces on $Q_t$ caused by $Q_1$ and $Q_1$ acting alone, or

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0|\vec{R}_1|^2}\mathbf{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0|\vec{R}_2|^2}\mathbf{a}_{R_2} + \cdots + \frac{Q_N}{4\pi\varepsilon_0|\vec{R}_N|^2}\mathbf{a}_{R_N}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{\boldsymbol{k=1}}^{\boldsymbol{N}}\frac{Q_{\boldsymbol{k}}\,(\vec{r} - \vec{r}_{\boldsymbol{k}})}{|\vec{r} - \vec{r}_{\boldsymbol{k}}|^{\boldsymbol{3}}}$$

***Example:*** *A charge of -0.3μC is located at* $(25,-30,15)$ *(in cm), and a second charge of 0.5μC is at* $(-10,8,12)$ *cm. Find* ***E*** *at: (a) the origin; (b)* $(15,20,50)$ *cm*

***Solution:***

$(a)\ \vec{\mathrm{E}} = \vec{\mathrm{E}}_1 + \vec{\mathrm{E}}_2$

$$\vec{\mathrm{E}}_1 = \frac{Q_1}{4\pi\varepsilon_0 |R_1|^2}\mathbf{a}_{R_1}$$

The point must be in meter $(25,-30,15)\ in\ cm = (0.25,-0.3,0.15) in\ m$

$(-10,8,12)\ in\ cm = (-0.1,0.08,0.12) in\ m$

$$\vec{R}_1 = (0,0,0) - (0.25,-0.3,0.15) = -0.25\boldsymbol{a}_x + 0.3\boldsymbol{a}_y - 0.15\boldsymbol{a}_z$$

$$\left|\vec{R}_1\right| = \sqrt{(0.25)^2 + (0.3)^2 + (0.15)^2} = 0.418$$

$$\vec{\mathrm{E}}_1 = \frac{-0.3\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.418)^2}\left[\frac{-0.25\boldsymbol{a}_x + 0.3\boldsymbol{a}_y - 0.15\boldsymbol{a}_z}{0.418}\right]$$

$$\vec{\mathrm{E}}_1 = 9233.77\boldsymbol{a}_x - 11080.5\boldsymbol{a}_y + 5540.26\boldsymbol{a}_z$$

$$\vec{R}_2 = (0,0,0) - (-0.1,-0.08,0.12) = 0.1\boldsymbol{a}_x + 0.08\boldsymbol{a}_y - 0.12\boldsymbol{a}_z$$

$$\left|\vec{R}_2\right| = \sqrt{(0.1)^2 + (0.08)^2 + (0.12)^2} = 0.175$$

$$\vec{\mathrm{E}}_2 = \frac{0.5\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.175)^2} * \frac{0.1\boldsymbol{a}_x + 0.08\boldsymbol{a}_y - 0.12\boldsymbol{a}_z}{0.175}$$

$$\vec{\mathrm{E}}_2 = 83888.55\boldsymbol{a}_x - 67110.8\boldsymbol{a}_y + 100666.26\boldsymbol{a}_z$$

$$\vec{\mathrm{E}} = \vec{\mathrm{E}}_1 + \vec{\mathrm{E}}_2 = 93122.3\boldsymbol{a}_x - 78190.52\boldsymbol{a}_y - 95126\boldsymbol{a}_z$$

$$\therefore \vec{\mathrm{E}} = \vec{\mathrm{E}}_1 + \vec{\mathrm{E}}_2 = 93.12\boldsymbol{a}_x - 78.19\boldsymbol{a}_y - 95.12\boldsymbol{a}_z\ KV/m$$

$(\boldsymbol{b})$

$E\ at(15,20,50)Cm? \quad \rightarrow \quad (0.15,0.2,0.5)m$

$R_1 = (0.15 - 0.25)\mathrm{a}_x + \big(0.2 - (-0.3)\big)\mathrm{a}_y + (0.5 - 0.15)\mathrm{a}_z$

$\therefore R_1 = -0.1\mathrm{a}_x + 0.5\mathrm{a}_y + 0.35\mathrm{a}_z$

$|R_1| = 0.618$

$R_2 = (0.15 - (-0.1))\mathrm{a}_x + (0.2 - 0.08)\mathrm{a}_y + (0.5 - 0.12)\mathrm{a}_z$

$R_2 = 0.25\mathrm{a}_x + 0.12\mathrm{a}_y + 0.38\mathrm{a}_z$

$|R_2| = 0.47$

$$E = E_1 + E_2 = \frac{Q_1}{4\pi\varepsilon_0 R_1}\,\mathrm{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0 {R_2}^2}\mathrm{a}_{R_2}$$

$$E = \frac{10^{-6}}{4\pi\varepsilon_0}\left[\frac{-0.3}{(0.618)^2} * \frac{-0.1\mathrm{a}_x + 0.5\mathrm{a}_y + 0.35\mathrm{a}_z}{0.618} + \frac{0.5}{(0.47)^2} \times \frac{0.25\mathrm{a}_x + 0.12\mathrm{a}_y + 0.38\mathrm{a}_z}{0.47}\right]$$

$$\therefore E = \frac{10^{-6}}{4\pi\varepsilon_0}\left[0.127\mathrm{a}_x - 0.635\mathrm{a}_y - 0.44\mathrm{a}_z + 1.2\mathrm{a}_x + 0.577\mathrm{a}_y + 1.83\mathrm{a}_z\right]$$

$$E = 11.9\mathrm{a}_x - 0.52\mathrm{a}_y + 12.4\mathrm{a}_z \ \ KV/m$$

***Example***: Point charges $1\ mC$ and $-2\ mC$ are located at $(3,2,-1)$ and $(-1,-1,4)$, respectively. Calculate the electric force on a $10\ nC$ charge located at $(0,3,1)$ and the electric field intensity at that point.

***Solution:***

$$\vec{\mathrm{F}} = \frac{Q}{4\pi\varepsilon_0}\sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \frac{Q_{\boldsymbol{k}}\,(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$$

$$\vec{r}-\vec{r}_1 = (0,3,1)-(3,2,-1)$$

$$= -3\boldsymbol{a}_x + \boldsymbol{a}_y + 2\boldsymbol{a}_z$$

$$|\vec{r}-\vec{r}_1| = \sqrt{(3)^2+(1)^2+(2)^2} = \sqrt{14}$$

$$\vec{r}-\vec{r}_{\boldsymbol{2}} = (0,3,1)-(-1,-1,4)$$

$$= \boldsymbol{a}_x + 4\boldsymbol{a}_y - 3\boldsymbol{a}_z$$

$$|\vec{r}-\vec{r}_{\boldsymbol{2}}| = \sqrt{(1)^2+(4)^2+(3)^2} = \sqrt{26}$$

$$\therefore\ \vec{\mathrm{F}} = \frac{10*10^{-\boldsymbol{9}}}{4\pi\,\frac{10^{-\boldsymbol{9}}}{36\pi}}\left\{\frac{1*10^{-\boldsymbol{3}}[-3\boldsymbol{a}_x+\boldsymbol{a}_y+2\boldsymbol{a}_z]}{\left|\sqrt{14}\right|^{\boldsymbol{3}}} + \frac{-2*10^{-\boldsymbol{3}}[\boldsymbol{a}_x+4\boldsymbol{a}_y-3\boldsymbol{a}_z)]}{\left|\sqrt{26}\right|^{\boldsymbol{3}}}\right\}$$

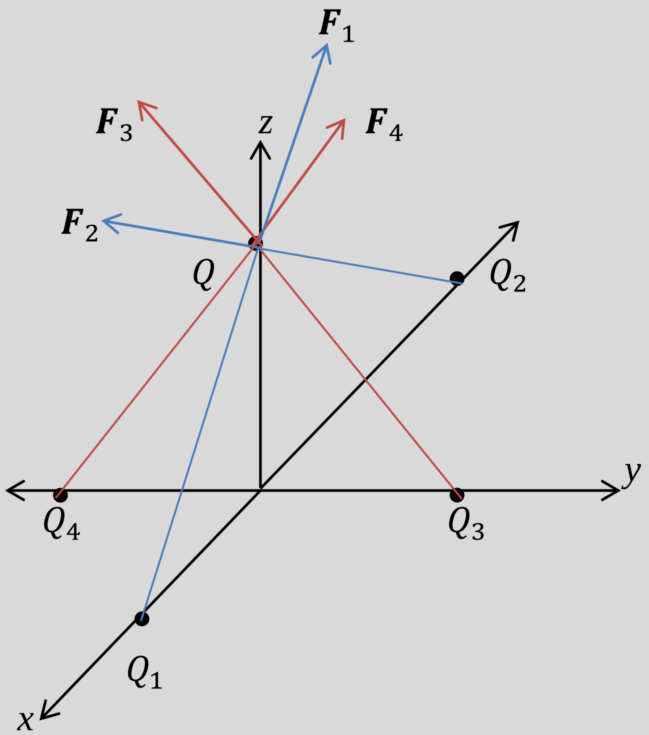
$$\vec{\mathrm{F}} = 90*10^{-\boldsymbol{3}}\left\{\frac{-3\boldsymbol{a}_x+\boldsymbol{a}_y+2\boldsymbol{a}_z}{\left|\sqrt{14}\right|^{\boldsymbol{3}}} + \frac{-2\boldsymbol{a}_x-8\boldsymbol{a}_y-6\boldsymbol{a}_z}{\left|\sqrt{26}\right|^{\boldsymbol{3}}}\right\}$$

$$\vec{\mathrm{F}} = -6.507\boldsymbol{a}_x - 3.817\boldsymbol{a}_y + 7.506\boldsymbol{a}_z\ mN$$

$$\vec{\mathrm{E}} = \frac{\vec{\mathrm{F}}_t}{Q_t} = \frac{(-6.507\boldsymbol{a}_x-3.817\boldsymbol{a}_y+7.506\boldsymbol{a}_z)}{10*10^{-\boldsymbol{9}}} = -650.7\boldsymbol{a}_x - 381.7\boldsymbol{a}_y + 750.6\boldsymbol{a}_z$$

***Example***: Four point charges each of 10 $\mu C$ are placed in free space at the point $(1,0,0), (-1,0,0), (0,1,0)$ and $(0,-1,0) m$ respectively. Determine the force on a point charge of 30 $\mu C$ located at a point $(0,0,1) m$

***Solution:***




$$\vec{F}_1 = \frac{Q Q_1 (\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_1|^3}$$

$$\vec{r} - \vec{r}_1 = (0,0,1) - (1,0,0)$$
$$= -\boldsymbol{a}_x + \boldsymbol{a}_z$$

$$|\vec{r} - \vec{r}_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\vec{F}_1 = \frac{30 * 10^{-6}}{4\pi \frac{10^{-9}}{36\pi}} \frac{* 10 * 10^{-6} (-\boldsymbol{a}_x + \boldsymbol{a}_z)}{\left|\sqrt{2}\right|^3}$$

$$\vec{F}_1 = 0.9533(-\boldsymbol{a}_x + \boldsymbol{a}_z)$$

$$\vec{F}_2 = \frac{QQ_2\,(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^{3}}$$

$$\vec{r}-\vec{r}_2 = (0,0,1)-(-1,0,0)$$
$$= \boldsymbol{a}_x+\boldsymbol{a}_z$$

$$|\vec{r}-\vec{r}_2| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$$

$$\vec{F}_2 = \frac{30*10^{-6}}{4\pi\frac{10^{-9}}{36\pi}}\frac{*\,10*10^{-6}(\boldsymbol{a}_x+\boldsymbol{a}_z)}{\left|\sqrt{2}\right|^{3}}$$

$$\vec{F}_2 = 0.9533(\boldsymbol{a}_x+\boldsymbol{a}_z)$$

$$\vec{F}_3 = \frac{QQ_3\,(\vec{r}-\vec{r}_3)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_3|^{3}}$$

$$\vec{r}-\vec{r}_3 = (0,0,1)-(0,1,0)$$
$$= -\boldsymbol{a}_y+\boldsymbol{a}_z$$

$$|\vec{r}-\vec{r}_3| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$$

$$\vec{F}_3 = \frac{30*10^{-6}}{4\pi\frac{10^{-9}}{36\pi}}\frac{*\,10*10^{-6}\left(-\boldsymbol{a}_y+\boldsymbol{a}_z\right)}{\left|\sqrt{2}\right|^{3}}$$

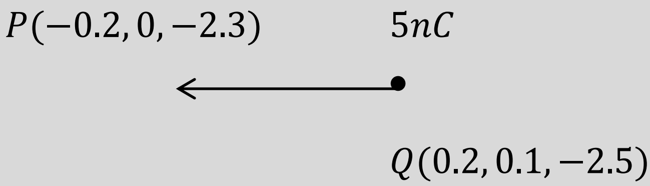
$$\vec{F}_3 = 0.9533\left(-\boldsymbol{a}_y+\boldsymbol{a}_z\right)$$

$$\vec{F}_4 = 0.9533\left(\boldsymbol{a}_y+\boldsymbol{a}_z\right)$$

$$\vec{F}_t = \vec{F}_1+\vec{F}_2+\vec{F}_3+\vec{F}_4 = 3.813\boldsymbol{a}_z \text{ N}$$

***Example***: Determine the electric field intensity at $P(-0.2, 0, -2.3)m$ due to a point charge of $5nC$ at $Q(0.2, 0.1, -2.5)m$ in air

***Solution:***




$$\vec{\mathrm{E}} = \frac{Q_1}{4\pi\varepsilon_0|R|^2}\mathbf{a}_{R_{1t}}$$

$$\vec{\mathrm{E}} = K\frac{Q}{R^2}\mathbf{a}_R$$

$$\vec{R} = (-0.2, 0, -2.3) - (0.2, 0.1, -2.5)$$
$$= -0.4\boldsymbol{a}_x - 0.1\boldsymbol{a}_y + 0.2\boldsymbol{a}_z$$

$$\left|\vec{R}\right| = \sqrt{(0.4)^2 + (0.1)^2 + (0.2)^2} = \sqrt{0.21} = 0.45$$

$$\mathbf{a}_R = \frac{\vec{R}}{\left|\vec{R}\right|}$$

$$\mathbf{a}_R = \frac{-0.4\boldsymbol{a}_x - 0.1\boldsymbol{a}_y + 0.2\boldsymbol{a}_z}{0.45}$$

$$\vec{\mathrm{E}} = K\frac{Q}{|R|^2}\mathbf{a}_R$$

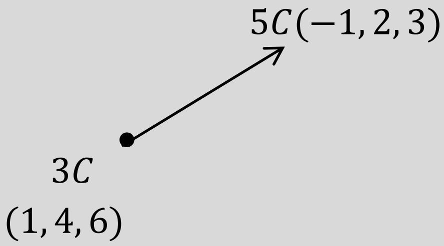
$$\vec{\mathrm{E}} = \frac{9 * 10^9 * 5 * 10^{-9}}{(0.45)^2}\left[\frac{-0.4\boldsymbol{a}_x - 0.1\boldsymbol{a}_y + 0.2\boldsymbol{a}_z}{0.45}\right]$$

$$\vec{\mathrm{E}} = -197.53\boldsymbol{a}_x - 49.38\boldsymbol{a}_y + 98.76\boldsymbol{a}_z \text{ V/m}$$

***Example:*** Find the force on $5C$ at $(-1,2,3)$ due to the point charge $3C$ at $(1,4,6)$

***Solution:***

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0 |R|^2}\mathbf{a}_{R_{1t}}$$




$$\vec{R} = (-1,2,3) - (1,4,6)$$
$$= -2\boldsymbol{a}_x - 2\boldsymbol{a}_y - 3\boldsymbol{a}_z$$

$$\left|\vec{R}\right| = \sqrt{(2)^2 + (2)^2 + (3)^2} = \sqrt{17}$$

$$\mathbf{a}_{R_{1t}} = \frac{\vec{R}}{\left|\vec{R}\right|}$$

$$\mathbf{a}_{R_{1t}} = \frac{-2\boldsymbol{a}_x - 2\boldsymbol{a}_y - 3\boldsymbol{a}_z}{\sqrt{17}}$$

$$\mathbf{F}_t = K\,\frac{Q_1 Q_t}{|R|^2}\mathbf{a}_{R_{1t}}$$

$$\vec{\mathrm{F}}_t = \frac{9*10^9*5*3}{\left(\sqrt{17}\right)^2}\left[\frac{-2\boldsymbol{a}_x - 2\boldsymbol{a}_y - 3\boldsymbol{a}_z}{\sqrt{17}}\right]$$

$$\vec{\mathrm{F}}_t = 7.9*10^9\left(\frac{-2}{\sqrt{17}}\boldsymbol{a}_x - \frac{2}{\sqrt{17}}\boldsymbol{a}_y - \frac{3}{\sqrt{17}}\boldsymbol{a}_z\right)$$

$$\vec{\mathrm{F}}_t = 3.8*10^9\boldsymbol{a}_x - 3.8*10^9\boldsymbol{a}_y - 5.74*10^9\boldsymbol{a}_z \text{ N}$$

## 3.5 Electric Fields Duo to Continuous Charge Distribution

So far we have only considered forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure 3.3.

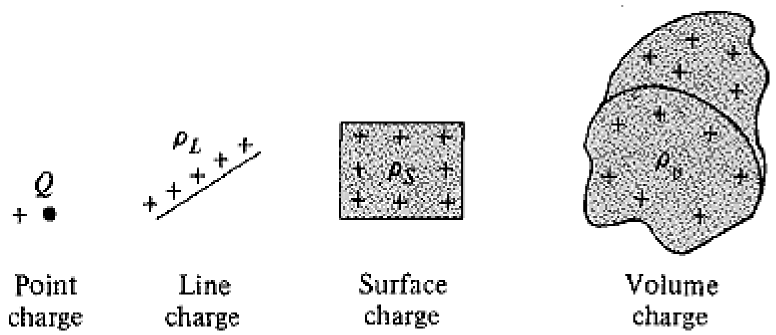



**Figure 3.3** Various charge distributions and charge elements.

It is customary to denote the

- ***line charge*** density by $\boldsymbol{\rho_L}$ (in $\mathbf{C/m}$),
- ***surface charge*** density by $\boldsymbol{\rho_S}$ (in $\mathbf{C/m^2}$), and
- ***volume charge*** density by $\boldsymbol{\rho_v}$ (in $\mathbf{C/m^3}$)

The ***electric field intensity*** due to each of the charge distributions $\rho_L$ , $\rho_s$ , and $\rho_v$ are given by

$$\vec{\mathrm{E}} = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\boldsymbol{line\ charge})$$

$$\vec{E} = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\boldsymbol{surface\ charge})$$

$$\vec{E} = \int_v \frac{\rho_v\, dv}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R \qquad (\boldsymbol{volume\ charge})$$

## a. Field of a Line Charge

Consider a line charge with uniform charge density $\rho_L$ extending from $A$ to $B$ along the z-axis as shown in Figure 3.4. The charge element $dQ$ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

the total charge $Q$ is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz$$

from Figure 3.4

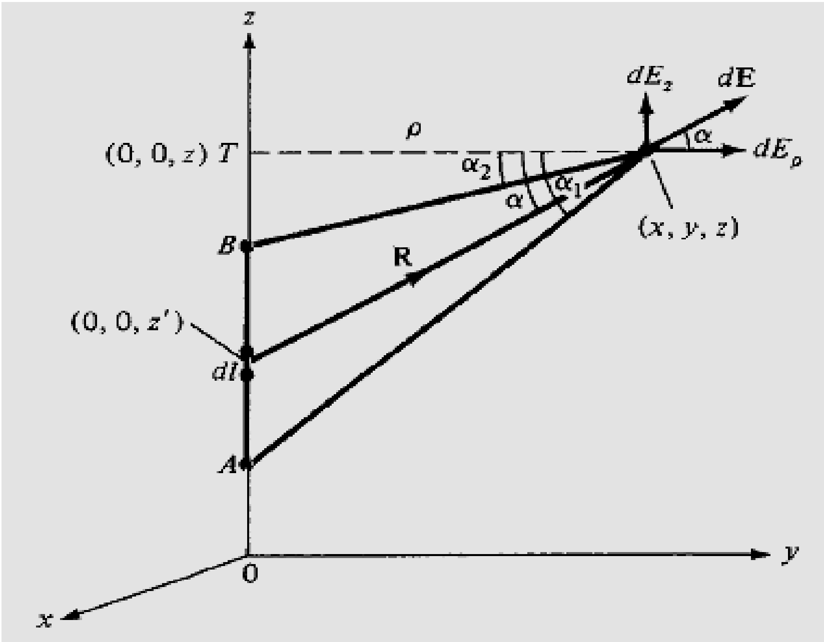



**Figure 3.4** Evaluation of the **E** field due to a line charge

$$dl = dz'$$

$$\vec{R} = (x, y, z) - (0, 0, z')$$

$$= x\boldsymbol{a}_x + y\boldsymbol{a}_y + (z - z')\boldsymbol{a}_z$$

or

$$\vec{R} = \rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z$$

$$R^2 = \left|\vec{R}\right|^2$$

$$= x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$R = \sqrt{\rho^2 + (z - z')^2}$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\vec{R}}{\left|\vec{R}\right|^3} = \frac{\rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{\sqrt{\rho^2 + (z - z')^2}\ [\rho^2 + (z - z')^2]^2} = \frac{\rho\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Substituting all this into eq. of ***electric field intensity*** we get

$$\vec{\mathrm{E}} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 |\vec{R}|^2} \mathbf{a}_R$$

$$\vec{\mathrm{E}} = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho \boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} \, dz' \qquad (*)$$

To evaluate this, it is convenient that we define $\alpha, \alpha_1$ and $\alpha_2$ as in Figure 3.4.

since $\quad \sec\alpha = \dfrac{R}{\rho} = \dfrac{\sqrt{\rho^2 + (z - z')^2}}{\rho}$

$$\therefore R = \sqrt{\rho^2 + (z - z')^2} = \rho \sec\alpha$$

since $\quad \tan\alpha = \dfrac{Tz'}{\rho} \qquad \therefore Tz' = \rho \tan\alpha$

$$z' = OT - \rho \tan\alpha \qquad dz' = -\rho \; sec^2\alpha \, d\alpha$$

$$\vec{\mathrm{E}} = \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \, sec^2\alpha \left[\cos\alpha \, \mathbf{a}_\rho + \sin\alpha \; \mathbf{a}_z\right] d\alpha}{\rho^2 \, sec^2 \, \alpha}$$

Hence, eq. (*) becomes

$$= \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \left[\cos\alpha \, \mathbf{a}_\rho + \sin\alpha \; \mathbf{a}_z\right] d\alpha$$

Thus for *a finite line charge,*

$$\vec{\mathrm{E}} = \frac{\rho_L}{4\pi\varepsilon_0\rho} \left[-(\sin\alpha_2 - \sin\alpha_1) \, \mathbf{a}_\rho + (\cos\alpha_2 - \cos\alpha_1) \, \mathbf{a}_z\right] \quad (**)$$

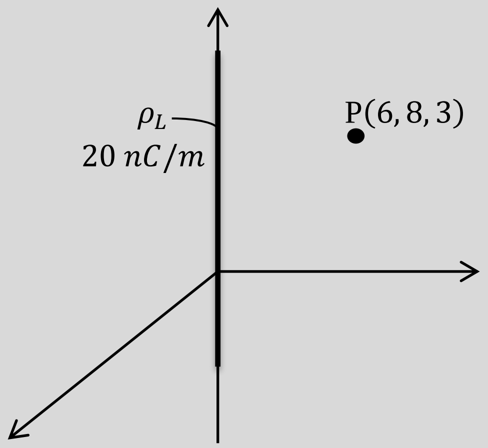
As a special case, for an *infinite line charge*, point *B* is at $(0,0,\infty)$ and *A* at $(0,0,-\infty)$ so that $\alpha_1 = \pi/2\,, \alpha_2 = -\pi/2$; the *z*-component vanishes and eq. (**) becomes

$$\vec{\mathrm{E}} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

***Example:*** A uniform line charge, infinite in extent with $\rho_L = 20\ nC/m$ lies along $z - axis$. Find the $\vec{\mathrm{E}}$ at $(6,8,3)m$ .

***Solution:***

$$\vec{\mathrm{E}} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$




$$\vec{\rho} = (6,8,3) - (0,0,3)$$

$$\vec{\rho}\ \ = 6\ \mathbf{a}_x + 8\ \mathbf{a}_y$$

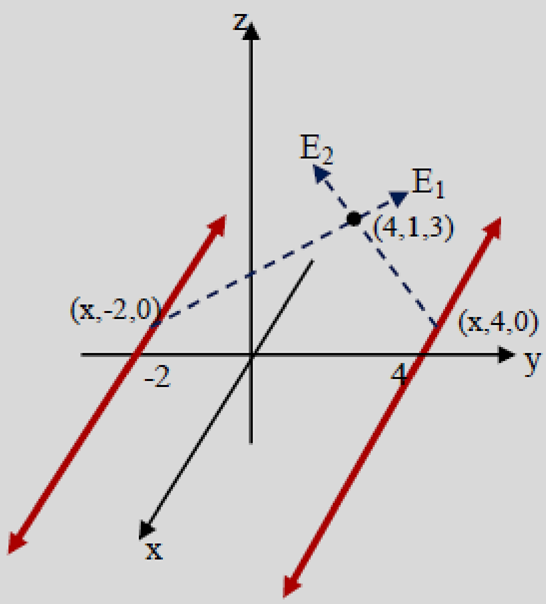
$$|\vec{\rho}| = \sqrt{6^2 + 8^2} = 10$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{6\ \mathbf{a}_x + 8\ \mathbf{a}_y}{10} = 0.6\ \mathbf{a}_x + 0.8\ \mathbf{a}_y$$

$$\vec{\mathrm{E}} = \frac{20 \times 10^{-9}}{2\pi \times 8.854 \times 10^{-12} \times 10}\left[0.6\ \mathbf{a}_x + 0.8\ \mathbf{a}_y\right] = 21.571\ \mathbf{a}_x + 28.761\ \mathbf{a}_y\ \mathrm{V/m}$$

***Example:*** Two uniform line charges of $\rho_L = 5\ nC/m$ each are parallel to the $x\ axis$, one at $z = 0, y = -2\ m$ and the other at $z = 0, y = 4m$. Find $\vec{E}$ at $(4, 1, 3)\ m$?

***Solution:***




$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$\vec{E}_1 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$$\vec{\rho}_1 = (1-(-2))\ \mathbf{a}_y + (3\ - 0)\ \mathbf{a}_z$$

$$\vec{\rho}_1\ \ = 3\ \mathbf{a}_y + 3\ \mathbf{a}_z$$

$$|\vec{\rho}_1| = \sqrt{3^2 + 3^2} = \sqrt{18}$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}_1}{|\vec{\rho}_1|} = \frac{3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{\sqrt{18}}$$

$$\vec{E}_1 = \frac{5\times10^{-9}}{2\pi\times8.8541\times10^{-12}}\left[\frac{3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{18}\right]\mathrm{V/m}$$

$$\vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$$\vec{\rho} = (1-4)\ \mathbf{a}_y + (3\ -0)\ \mathbf{a}_z$$

$$\vec{\rho}\ \ = -3\ \mathbf{a}_y + 3\ \mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{3^2+3^2} = \sqrt{18}$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{-3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{\sqrt{18}}$$

$$\vec{E}_2 = \frac{5\times10^{-9}}{2\pi\times8.8541\times0^{-12}}\left[\frac{-3\ \mathbf{a}_y + 3\ \mathbf{a}_z}{18}\right]\ \mathrm{V/m}$$

$$\vec{E} = 2 * \frac{5\times10^{-9}}{2\pi\times8.8541\times10^{-12}}\ \frac{6\ \mathbf{a}_z}{18} = 30\ \mathbf{a}_z\ \mathrm{V/m}$$

## b. Field of a Sheet Charge

Consider an infinite sheet of charge in the *xy*-plane with uniform charge density $\rho_S$. The charge associated with an elemental area $dS$ is

$$dQ = \rho_S\ dS$$

from the eq.

$$\vec{E} = \int_S \frac{\rho_S\ dS}{4\pi\varepsilon_0\left|\vec{R}\right|^2}\mathbf{a}_R$$

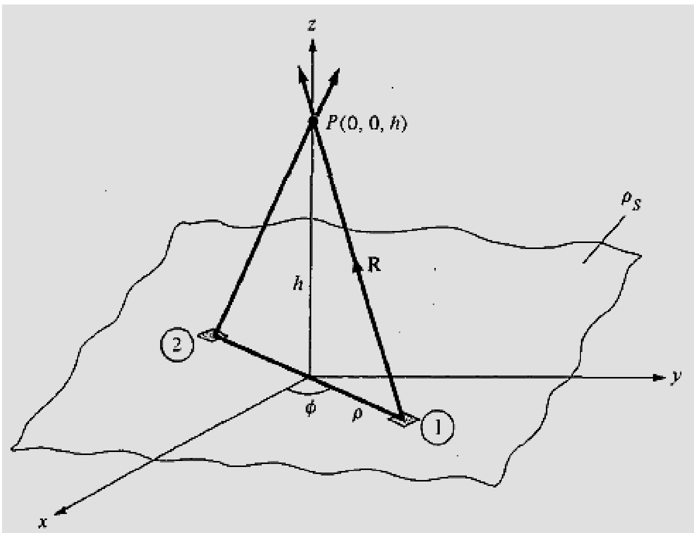



**Figure 3.5** Evaluation of the E field due to an infinite sheet of charge.

From eq. above, the contribution to the $\vec{E}$ field at point P(0, 0, $h$) by the elemental surface 1 shown in Figure 3.5 is

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 |\vec{R}|^2}\,\mathbf{a}_R (***)$$

From Figure 3.5,

$$\vec{R} = \rho\,(-\mathbf{a}_\rho) + h\,\mathbf{a}_z$$

$$= -\mathbf{a}_\rho \rho + h\,\mathbf{a}_z$$

$$|\vec{R}| = [\rho^2 + h^2]^{1/2}$$

$$\mathbf{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z}{[\rho^2 + h^2]^{1/2}}$$

$$dQ = \rho_S\, dS = \rho_S\, \rho\, d\emptyset\, d\rho$$

substitution of these terms into eq. (***) gives

$$d\vec{E} = \frac{\rho_S\, \rho\, d\emptyset\, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]} \left[\frac{-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z}{[\rho^2 + h^2]^{1/2}}\right]$$

$$= \frac{\rho_S\, \rho\, d\emptyset\, d\rho\, [-\mathbf{a}_\rho \rho + h\,\mathbf{a}_z]}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}$$

$$d\vec{E} = d\vec{E}_\rho + d\vec{E}_z$$

Since $d\vec{E}_\rho = 0$ from the symmetry of the charge distribution,

$$d\vec{E} = \frac{\rho_S\, h\, \rho\, d\emptyset\, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}\,\mathbf{a}_z$$

$$\vec{E} = \int_S d\vec{E}_z = \frac{\rho_S}{4\pi\varepsilon_0} \int_{\emptyset=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\, \rho\, d\emptyset\, d\rho}{[\rho^2 + h^2]^{3/2}}\,\mathbf{a}_z$$

$$\vec{E} = \frac{\rho_S\, h}{4\pi\varepsilon_0}\, 2\pi \int_{\rho=0}^{\infty} [\rho^2 + h^2]^{-3/2}\, \frac{1}{2}\, d(\rho^2)\, \mathbf{a}_z$$

$$= \frac{\rho_S\, h}{2\varepsilon_0}\{-[\rho^2 + h^2]^{-1/2}\}_0^{\infty}\, \mathbf{a}_z$$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\, \mathbf{a}_z$$

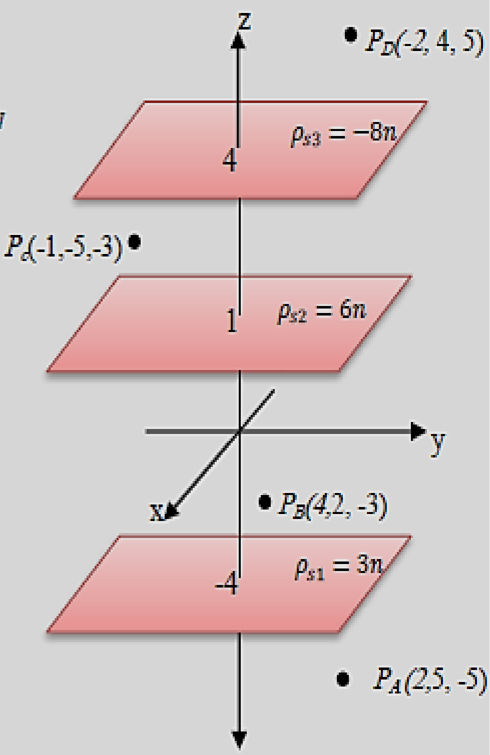
for an *infinite sheet* of charge

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\, \mathbf{a}_N$$

***Example:*** Three infinite uniform sheets of charge are located in free space as follows: 3nC/ m$^2$ at z = -4, 6nC/m$^2$ at $z$ = 1, and - 8nC/m$^2$ at $z$ = 4. Find E at the point: (a) $P_A(2, 5, -5)$; (b) $P_B(4, 2, -3)$; (c) $P_c(-1, -5, 2)$; (d) $P_D(-2, 4, 5)$?

***Solution:***

$a -$ $\quad at\ p_A$




$$Becuse\ the\ infinite\ sheet\ charge\ the\ E = \vec{E} = \frac{\rho_S}{2\varepsilon_o}\mathrm{a}_N$$

$$E_T = \left[\frac{-3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = -56.5\ \mathrm{a}_z\ V/m$$

b- $\quad at\ p_B$

$$E = \left[\frac{3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 282.3\ \mathrm{a}_z\ V/m$$

c- $\quad at\ p_C$

$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 960.45\mathrm{a}_z\ V/m$$

$d -$ $\quad at\ p_d$

$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} - \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 56.5\mathrm{a}_z\ V/m$$

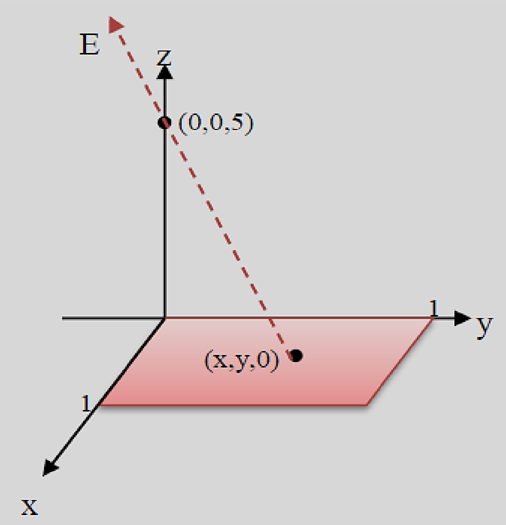
***Example:*** The finite sheet $0 \le x \le 1, 0 \le y \le 1$ on the $z = 0$ plane has a charge density $\rho_S = xy(x^2 + y^2 + 25)^{\frac{3}{2}}\, nC/m^2$. Find

a. The total charge on the sheet

b. The electric field $(\vec{E})$ at $(0, 0, 5)$?

c. The force experienced by a $-1nc$ charge located at $(0, 0, 5)$?

***Solution:***




a) $Q = \int_S \rho_S \, dS$

$$Q = \int_{y=0}^{1} \int_{x=0}^{1} xy(x^2 + y^2 + 25)^{\frac{3}{2}} \; dx\, dy$$

$$Q = \left\{ \frac{1}{2} \cdot \frac{2}{5} \int_{y=0}^{1} \left[ y(x^2 + y^2 + 25)^{\frac{5}{2}} \right]_0^1 dy \right\}$$

$$Q = \left\{ \frac{1}{2} \cdot \frac{2}{5} \int_{y=0}^{1} y \left[ (y^2 + 26)^{\frac{5}{2}} - (y^2 + 25)^{\frac{5}{2}} \right] dy \right\}$$

$$Q = \left\{ \frac{1}{2} \cdot \frac{2}{5} \int_{y=0}^{1} y \left[ (y^2 + 26)^{\frac{5}{2}} - (y^2 + 25)^{\frac{5}{2}} \right] dy \right\}$$

$$Q = \left\{ \frac{1}{5} \cdot \frac{1}{7} \left[ (y^2 + 26)^{\frac{7}{2}} - (y^2 + 25)^{\frac{7}{2}} \right]_0^1 \right\}$$

$$Q = \frac{1}{35} \left[ (27)^{\frac{7}{2}} + (25)^{\frac{7}{2}} - 2(26)^{\frac{7}{2}} \right] = 33.15\, nC$$

$$b)\quad \vec{E} = \int \frac{\rho_S\, ds}{4\pi\varepsilon_0 \left|\vec{R}\right|^2}\,\mathbf{a}_R$$

$$dS = dx\, dy$$

$$\vec{R} = (0,0,5) - (x,y,0) = -x\,\mathbf{a}_x - y\,\mathbf{a}_y + 5\,\mathbf{a}_z$$

$$\left|\vec{R}\right| = \sqrt{x^2 + y^2 + 25}$$

$$\mathbf{a}_R = \frac{\vec{R}}{\left|\vec{R}\right|} = \frac{-x\,\mathbf{a}_x - y\,\mathbf{a}_y + 5\,\mathbf{a}_z}{\sqrt{x^2+y^2+25}}$$

$$\vec{E} = \int_{y=0}^{1}\int_{x=0}^{1} \frac{xy(x^2+y^2+25)^{\frac{3}{2}}}{4\pi\varepsilon_0\left(\sqrt{x^2+y^2+25}\right)^2} * \frac{-x\,\mathbf{a}_x - y\,\mathbf{a}_y + 5\,\mathbf{a}_z}{\sqrt{x^2+y^2+25}}\quad dx\,dy$$

$$\vec{E} = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\int_{y=0}^{1}\int_{x=0}^{1} \frac{xy(x^2+y^2+25)^{\frac{3}{2}}}{(x^2+y^2+25)^{\frac{3}{2}}}\left(-x\,\mathbf{a}_x - y\,\mathbf{a}_y + 5\,\mathbf{a}_z\right)\quad dx\,dy$$

$$\vec{E} = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\int_{y=0}^{1}\int_{x=0}^{1} xy\ \left(-x\,\mathbf{a}_x - y\,\mathbf{a}_y + 5\,\mathbf{a}_z\right) dx\,dy$$

$$\vec{E} = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\left[-\int_{y=0}^{1}\int_{x=0}^{1} x^2y\,dx\,dy\,\mathbf{a}_x - \int_{y=0}^{1}\int_{x=0}^{1} xy^2dx\,dy\,\mathbf{a}_y + 5\int_{y=0}^{1}\int_{x=0}^{1} xy\,dx\,dy\ \mathbf{a}_z\right]$$

$$\vec{E} = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\left[-\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 \boldsymbol{a}_x - \left[\frac{x^2}{2}\right]_0^1\left[\frac{y^3}{3}\right]_0^1 \boldsymbol{a}_y + 5\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 \boldsymbol{a}_z\right]$$

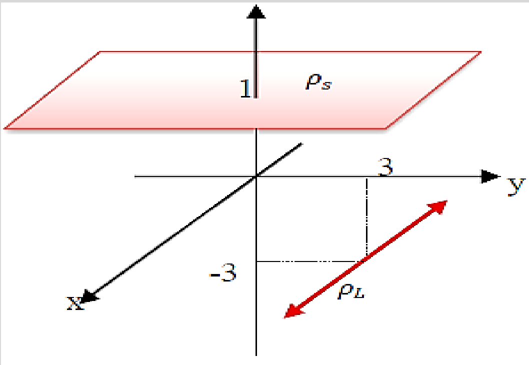
$$\vec{E} = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\left[-\frac{1}{3}\frac{1}{2}\,\boldsymbol{a}_x - \frac{1}{2}\frac{1}{3}\,\boldsymbol{a}_y + 5\frac{1}{2}\frac{1}{2}\,\boldsymbol{a}_z\right] = \frac{1\times 10^{-9}}{4\pi\varepsilon_0}\left[-\frac{1}{6}\,\boldsymbol{a}_x - \frac{1}{6}\,\boldsymbol{a}_y + \frac{5}{4}\,\boldsymbol{a}_z\right]$$

$$\vec{E} = -1.5\,\boldsymbol{a}_x - 1.5\,\boldsymbol{a}_y + 11.23\,\boldsymbol{a}_z\ \ V/m$$

c. $\vec{F} = Q\vec{E} = -1\times 10^{-9}\left(-1.5\,\boldsymbol{a}_x - 1.5\,\boldsymbol{a}_y + 11.23\,\boldsymbol{a}_z\right) N$

***Example:*** A uniform sheet charge with $\rho_S = 1/3\pi\ nC/m^2$ is located at $z = 5\ m$ and a uniform line charge with $\rho_L = 25/9\ nC/m$ at $y = 3m \quad and \quad z = -3\ m$. Find $\vec{E}$ at $(x, -1, 0)m$

***Solution:***




$$\vec{E}_T = \vec{E}_1 + \vec{E}_2$$

$\vec{E}_1$ *due to surface charge*

$\vec{E}_2$ *due to line charge*

$$\vec{E}_1 = \frac{\rho_S}{2\varepsilon_0}\ \mathbf{a}_N$$

$$\vec{E}_1 = \frac{(1/3\pi) \times 10^{-9}}{2\varepsilon_0}(-\,\boldsymbol{a}_z) = -6\ \boldsymbol{a}_z$$

$$\vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$$\vec{\rho} = (x, -1, 0) - (x, 3, -3) = (-1 - 3)\ \mathbf{a}_y + (0\ - (-3))\ \mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{-4\ \boldsymbol{a}_y + 3\ \boldsymbol{a}_z}{5}$$

$$\vec{E}_2 = \frac{(25/9\,) \times 10^{-9}}{2\pi \times 8.854 \times 10^{-12} \times 5}\left[\frac{-4\ \boldsymbol{a}_y + 3\ \boldsymbol{a}_z}{5}\right]$$

$$\vec{E}_2 = 8\ \boldsymbol{a}_y + 6\ \boldsymbol{a}_z$$

$$\vec{E}_T = -6\ \boldsymbol{a}_z + 8\ \boldsymbol{a}_y + 6\ \boldsymbol{a}_z = 8\ \boldsymbol{a}_y \qquad V/m$$

***Example:*** A circular ring of radius $a$ carries a uniform charge $pL\ C/m$ and is placed on the $xy-plane$ with axis the same as the $z-axis$.

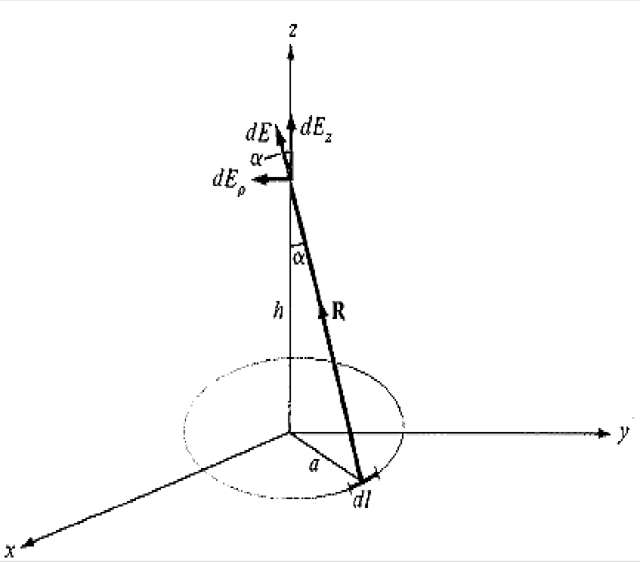
(a) Show that

$$\boldsymbol{E}(0,0,h)=\frac{\rho_L\, ah}{2\varepsilon_0[h^2+a^2]^{\frac{3}{2}}}\ \boldsymbol{a}_z$$

(b) What values of $h$ gives the maximum value of $\boldsymbol{E}$?

(c) If the total charge on the ring is $Q$, find $\boldsymbol{E}$ as $a\ \rightarrow\ 0$.

***Solution:***




(a). $d\vec{L}=d\rho\ \boldsymbol{a_\rho}+\rho\ d\emptyset\ \boldsymbol{a}_\emptyset+dz\ \boldsymbol{a_z}$

From the figure $\rho=a$ $\therefore dl=a\ d\emptyset$

$\vec{R}+a\ \boldsymbol{a_\rho}=h\ \boldsymbol{a_z}\quad\rightarrow\quad\vec{R}=-a\ \boldsymbol{a_\rho}+h\ \boldsymbol{a_z}$

$$R=\left|\vec{R}\right|=\sqrt{a^2+h^2}\quad,\quad\boldsymbol{a}_R=\frac{\vec{R}}{\left|\vec{R}\right|}\quad,\quad\frac{\boldsymbol{a}_R}{\left|\vec{R}\right|^2}=\frac{\vec{R}}{\left|\vec{R}\right|^3}=\frac{-a\ \boldsymbol{a_\rho}+h\ \boldsymbol{a_z}}{[a^2+h^2]^{3/2}}$$

$$\vec{E}=\int_L\frac{\rho_L\ dl}{4\pi\varepsilon_0\left|\vec{R}\right|^2}\boldsymbol{a}_R$$

$$\vec{E}=\frac{\rho_L}{4\pi\varepsilon_0}\int_{\emptyset=0}^{2\pi}\frac{\left(-a\ \boldsymbol{a_\rho}+h\ \boldsymbol{a_z}\right)}{[a^2+h^2]^{3/2}}\ a\ d\emptyset$$

*By symmetry, the contributions along* $\boldsymbol{a_\rho}$ *add up to zero.*

$$\vec{E} = \frac{\rho_L\, a\, h\, \boldsymbol{a}_z}{4\pi\varepsilon_0[a^2+h^2]^{3/2}}\int_{\emptyset=0}^{2\pi} d\emptyset \;= \frac{\rho_L\, a\, h\, \boldsymbol{a}_z}{2\varepsilon_0[a^2+h^2]^{3/2}}$$

(b).

$$\frac{d|\vec{E}|}{dh} = \frac{\rho_L\, a}{2\varepsilon_0}\left\{\frac{[a^2+h^2]^{3/2}(1) - \frac{3}{2}2h^2[a^2+h^2]^{1/2}}{[a^2+h^2]^3}\right\}$$

For maximum $\vec{E}$, $\frac{d|\vec{E}|}{dh} = 0$, which implies that

$$[a^2+h^2]^{1/2}\;[a^2+h^2-3h^2] = 0$$

$$a^2 - 2h^2 = 0 \quad or \quad h = \pm\frac{a}{\sqrt{2}}$$

(c). Since the charge is uniformly distributed, the line charge density is

$$\rho_L = \frac{Q}{2\pi a}$$

so that

$$\vec{E} = \frac{Q\, h}{4\pi\varepsilon_0[a^2+h^2]^{3/2}}\;\boldsymbol{a}_z$$

As $a \to 0$

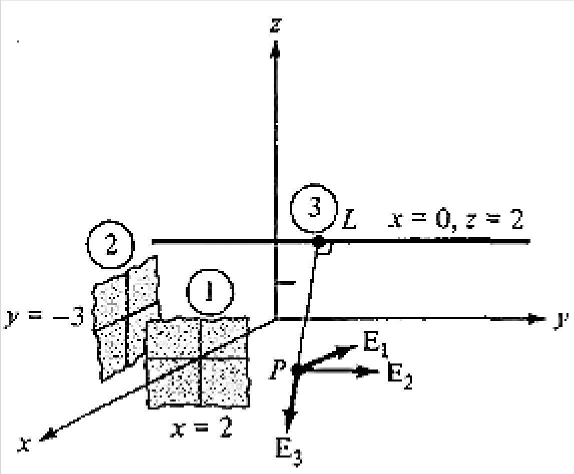
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 h^2}\;\boldsymbol{a}_z$$

or in general

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\;\boldsymbol{a}_R$$

***Example:*** Planes $x = 2$ and $y = -3$, respectively, carry charges $10\ nC/m^2$ and $15\ nC/m^2$. If the line $x = 0, z = 2$ carries charge $10\pi\ nC/m$, calculate $\boldsymbol{E}$ at $(1,1,-1)$ due to the three charge distributions.

***Solution:***




$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_0}\ \mathbf{a}_N$$

$$\vec{E}_1 = \frac{\rho_{S_1}}{2\varepsilon_0}\ (-\boldsymbol{a}_x) = -\frac{10 \cdot 10^{-9}}{2 \cdot \frac{10^{-9}}{36\pi}}\ \boldsymbol{a}_x = -180\pi\ \ \boldsymbol{a}_x$$

$$\vec{E}_2 = \frac{\rho_{S_2}}{2\varepsilon_0}\ \boldsymbol{a}_y = \frac{15 \cdot 10^{-9}}{2 \cdot \frac{10^{-9}}{36\pi}}\ \boldsymbol{a}_y = 270\pi\ \ \boldsymbol{a}_y$$

$$\vec{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\ \mathbf{a}_\rho$$

$$\vec{\rho}\ \ = (1,1,-1) - (0,1,2) = \ \mathbf{a}_x - 3\ \mathbf{a}_z$$

$$|\vec{\rho}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\mathbf{a}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|} = \frac{\boldsymbol{a}_x - 3\,\boldsymbol{a}_z}{\sqrt{10}}$$

$$\vec{E}_3 = \frac{10\pi \cdot 10^{-9}}{2\pi \cdot \frac{10^{-9}}{36\pi}\,\sqrt{10}} \cdot \frac{\boldsymbol{a}_x - 3\,\boldsymbol{a}_z}{\sqrt{10}} = 18\pi(\,\boldsymbol{a}_x - 3\,\boldsymbol{a}_z)$$

$$\vec{E} = -180\pi\;\boldsymbol{a}_x + 270\pi\;\boldsymbol{a}_y + 18\pi(\,\boldsymbol{a}_x - 3\,\boldsymbol{a}_z)$$

$$= -162\pi\;\boldsymbol{a}_x + 270\pi\;\boldsymbol{a}_y - 54\pi\,\boldsymbol{a}_z \quad V/m$$

## a. Field Due to a Continuous Volume Charge Distribution

If we now visualize a region of space filled with a great number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a ***volume charge density*** $\rho_v\;C/m^3$

إذا تصورنا منطقة من الفراغ مملؤة بعدد هائل من الشحنات المنفصلة عن بعضها بمسافات صغيرة جدا، فأننا نستطيع أحلال هذ التوزيع لجسيمات صغيرة بتوزيع أملس يوصف بكثافة شحنة حجمية.

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{vol} \rho_v\; dv$$

$$\vec{E} = \int_{vol} \frac{\rho_v\; dv}{4\pi\varepsilon_0 \left|\vec{R}\right|^2}\,\boldsymbol{a}_R$$

***Example:*** Calculate the total charge within each of the indicate volumes:

(a) $0.1 \le x, y, z \le 0.2$ $\rho_v = \frac{1}{x^3 y^3}$

(b) $0 \le \rho \le 0.1$ $0 \le \emptyset \le \pi$ $2 \le z \le 4$ $\rho_v = \rho^2 z^2 \sin 0.6\emptyset$

(c) $Univers$ $\rho_v = \frac{e^{-2r}}{r^2}$

***Solution:***

(a)

$$Q = \int_{vol} \rho_v \, dv \qquad dv = dx\,dy\,dz$$

$$Q = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \int_{x=0.1}^{0.2} \frac{1}{x^3 y^3}\, dx\,dy\,dz = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \int_{x=0.1}^{0.2} x^{-3} y^{-3}\, dx\,dy\,dz$$

$$Q = \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \frac{1}{y^3}\, dy\,dz$$

$$= \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \int_{z=0.1}^{0.2} \int_{y=0.1}^{0.2} \frac{1}{y^3}\, dy\,dz$$

$$= \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \left[-\frac{1}{2y^2}\right]_{0.1}^{0.2} \int_{z=0.1}^{0.2} dz = \left[-\frac{1}{2x^2}\right]_{0.1}^{0.2} \left[-\frac{1}{2y^2}\right]_{0.1}^{0.2} [z]_{0.1}^{0.2}$$

$$= \left[-\frac{1}{2(0.2)^2} + \frac{1}{2(0.1)^2}\right] \left[-\frac{1}{2(0.2)^2} + \frac{1}{2(0.1)^2}\right] [0.2 - 0.1]$$

$$Q = 140.6 \ C$$

(b)

$$Q = \int_{vol} \rho_v \, dv \qquad\qquad dv = \rho \, d\rho \, d\emptyset \, dz$$

$$Q = \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \int_{\rho=0}^{0.1} \rho^2 z^2 \sin 0.6\emptyset \;\; \rho \, d\rho \, d\emptyset \, dz$$

$$= \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \int_{\rho=0}^{0.1} \rho^3 z^2 \sin 0.6\emptyset \;\; d\rho \, d\emptyset \, dz$$

$$= \int_{z=2}^{4} \int_{\emptyset=0}^{\pi} \left[\frac{\rho^4}{4}\right]_0^{0.1} z^2 \sin 0.6\emptyset \;\; d\emptyset \, dz = \int_{z=2}^{4} \left[\frac{\rho^4}{4}\right]_0^{0.1} \left[\frac{-\cos 0.6\emptyset}{0.6}\right]_0^{\pi} z^2 \, dz$$

$$= \left[\frac{\rho^4}{4}\right]_0^{0.1} \left[\frac{-\cos 0.6\emptyset}{0.6}\right]_0^{\pi} \left[\frac{z^3}{3}\right]_2^{4} = 1.018 \;\; mC$$

(c)

$$Q = \int_{vol} \rho_v \, dv \qquad\qquad dv = r^2 \sin\theta \;\, dr \, d\theta \, d\emptyset$$

$$Q = \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \frac{e^{-2r}}{r^2} \;\; r^2 \sin\theta \;\, dr \, d\theta \, d\emptyset$$

$$= \left[-\frac{1}{2} e^{-2r}\right]_0^{\infty} [-\cos\theta]_0^{\pi} \, [\emptyset]_0^{2\pi} \; = \left[-\frac{1}{2} e^{-\infty} + \frac{1}{2} e^{0}\right] [1+1] \, [2\pi]$$

$$= 6.28 \;\;\; C$$

## Home Work

1. A charge is distributed on y-axis of Cartesian system having a line charge density of $(5\,y^{3.5})\,\mu C/m$. Find the total charge over length of $15\,m$.

   Ans: $0.2178\,C$

2. A charge of $10\,C$ is located at the point $x = 0$ and $y = 1$ and charge of $-5\,C$ is at the point $x = 0$ and $y = -1$. Find the point on $y - axis$ at which let $\vec{E} = 0$.

   Ans: $(0, -5.828)$ $or$ $(0, -0.1716)$

3. A point charge of $20\,nC$ is located at the origin. Determine the magnitude of $\vec{E}$ at point $P(1, 3, -4)m$.

   Ans: $\vec{E} = 1.357\ \boldsymbol{a}_x + 4.073\ \boldsymbol{a}_y - 54\pi\,\boldsymbol{a}_z \frac{V}{m}$

4. On the line $x = 4$ and $y = -4$, there is a uniform charge distribution with density $\rho_L = 25\ \frac{nC}{m}$. Determine $\vec{E}$ at $(-2, -1, 4)m$.

   Ans: $\vec{E} = -59.92\,\boldsymbol{a}_x + 29.969\ \boldsymbol{a}_y\ \ V/m$

5. Four infinity sheets of charges with uniform charges density $20\ \frac{PC}{m^2}, -8\ \frac{PC}{m^2}, 6\ \frac{PC}{m^2}$ and $-180\ \frac{PC}{m^2}$ are located at $y = 6, y = 2,$ $y = -2$ $\quad and \quad$ $y = -5$ respectively. Find $\vec{E}$ at
   a) $(2, 5, -6)$
   b) $(0, 0, 0)$
   c) $(-1, -2.1, 6)$
   d) $(10^6, 10^6, 10^7)$ Ans: $(-2.26\ \boldsymbol{a}_y\ \frac{V}{m}, -1.3555\ \boldsymbol{a}_y\ \frac{V}{m}, -2.03\ \boldsymbol{a}_y\ \frac{V}{m}, 0\ \frac{V}{m})$

## 3.5 Electric Flux Density كثافة الفيض الكهربائي

**Michael Faraday** had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (dielectric material*)* which would occupy the entire volume between the concentric spheres

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

**Faraday** found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as displacement, displacement flux, or simply electric flux.

**Faraday's** experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere

$$\Psi = Q$$

Where $\Psi$ (psi) is electric flux in coulombs C

We can obtain more quantitative information by considering an inner sphere of radius $a$ and an outer sphere of radius b, with charges of $Q$ and $-Q$, respectively (Fig. 3.6). The paths of

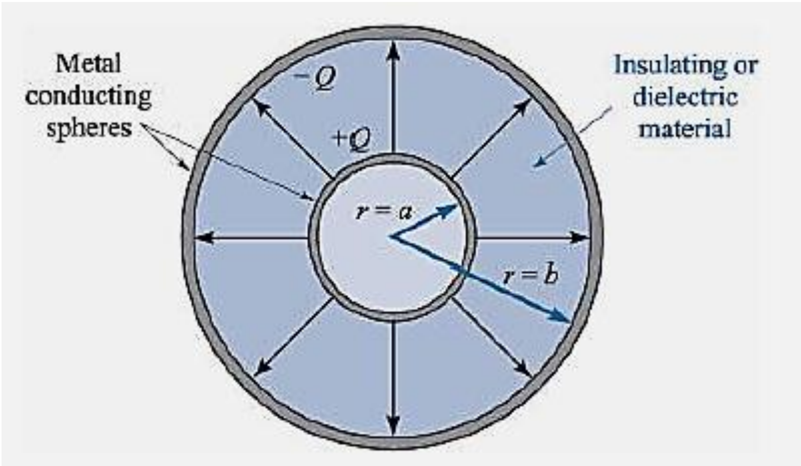



**Figure 3.6:** The electric flux in the region between a pair of charged concentric sphere

Electric flux $\Psi$ extending from the inner sphere to the outer sphere is indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, $\Psi$ coulombs of electric flux are produced by the charge $Q(=\Psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2 m^2$ . The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2\ C/m^2$, and this is an important new quantity.

Referring again to Fig. 3.6, the electric flux density is in the radial direction and has a value of

$$\vec{D}(at\ r = a) = \frac{Q}{4\pi a^2}\ \boldsymbol{a}_r \qquad (inner\ shere)$$

$$\vec{D}(at\ r = b) = \frac{Q}{4\pi b^2}\ \boldsymbol{a}_r \qquad (outer\ shere)$$

and at a radial distance $r$, where $a \leq r \leq b$

$$\vec{D} = \frac{Q}{4\pi r^2}\ \boldsymbol{a}_r$$

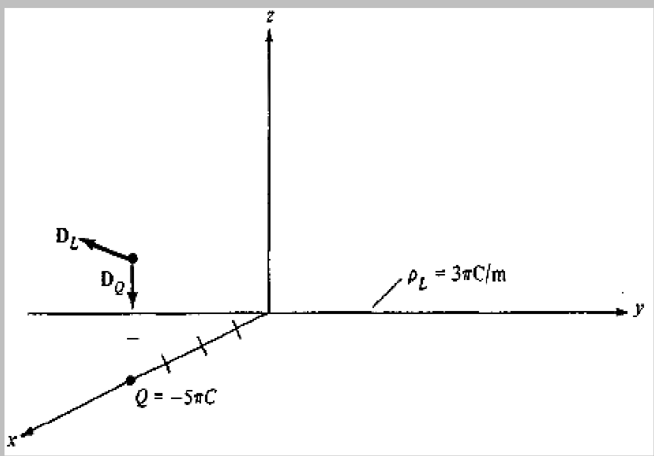
If we now let the inner sphere become smaller and smaller, while still retaining a charge of $Q$, it becomes a point charge in the limit, but the electric flux density at a point *r* meters from the point charge is still given by

$$\vec{D} = \frac{Q}{4\pi r^2}\ \boldsymbol{a}_r$$

$$\vec{D} = \varepsilon_0 \vec{E} \qquad (in\ free\ space)$$

***Example***: Determine $\vec{D}$ at $(4, 0, 3)$ if there is a point charge $-5\pi\ mC$ at $(4, 0, 0)$ and a line charge $3\pi\ mC/m$ along the $y - axis$.

***Solution:***




$$\vec{D} = \vec{D}_Q + \vec{D}_L$$

$\vec{D}_Q \quad due\ to\ the\ point\ charge$

$\vec{D}_L \quad due\ to\ the\ line\ charge$

$$\vec{D}_Q = \varepsilon_0 \vec{E}$$

$$\vec{D}_Q = \frac{Q}{4\pi\left|\vec{R}\right|^2}\ \boldsymbol{a}_R$$

$$\vec{R} = (4, 0, 3) - (4, 0, 0)$$

$$\vec{R} = 3\ \mathbf{a}_z \qquad \left|\vec{R}\right| = \sqrt{3^2} = 3$$

$$\mathbf{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{3\,\boldsymbol{a}_z}{3} = \boldsymbol{a}_z$$

$$\vec{D}_Q = \frac{-5\pi * 10^{-3}}{4\pi * 3^2}\,\boldsymbol{a}_z = -0.138\,\boldsymbol{a}_z \quad mC/m^2$$

$$\vec{D}_L = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho$$

$$\rho = (4,0,3) - (0,0,0) = 4\,\boldsymbol{a}_x + 3\,\boldsymbol{a}_z$$

$$|\,\vec{\rho}| = \sqrt{4^2+3^2} = 5$$

$$\vec{D}_L = \frac{3\pi}{2\pi * 5}\,\frac{4\,\boldsymbol{a}_x + 3\,\boldsymbol{a}_z}{5} = 0.24\,\boldsymbol{a}_x + 0.18\,\boldsymbol{a}_z \;\; mC/m^2$$

Since

$$\vec{D} = \vec{D}_Q + \vec{D}_L$$

$$\therefore\ \vec{D} = -0.138\,\boldsymbol{a}_z + 0.24\,\boldsymbol{a}_x + 0.18\,\boldsymbol{a}_z$$

$$= 240\,\boldsymbol{a}_x + 42\,\boldsymbol{a}_z \quad \mu C/m^2$$

## 3.6 Gauss's Law

These generalizations of Faraday's experiment lead to the following statement, which is known as **Gauss's law**:

***"The electric flux passing through any closed surface is equal to the total charge enclosed by that surface"***

**قانون جاوس" : الفيض الكهربائي المار خلال اي سطح مغلق يساوي الشحنة الكمية المحتواة بذلك السطح"**

$$\Psi = Q_{enclosed}$$

$$\Delta\Psi = \vec{D}_S \cdot \Delta S$$

$$\Psi = \oint_S \vec{D}_S \cdot d\vec{S}$$

توضع دائرة صغيرة على علامة التكامل لتشير الى ان التكامل مؤدى على سطح مغلق ويسمى هذا السطح ب "سطح جاوس"

- ***The charge enclosed might be several point charges, in which case***

$$Q_{enclosed} = \sum Q_n$$

- ***or a line charge***

$$Q_{enclosed} = \int \rho_L \, dL$$

- ***or a surface charge***

$$Q_{enclosed} = \int_S \rho_S \, dS$$

- ***or a volume charge***

$$Q_{enclosed} = \int_{vol} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding Gauss's law may be written in terms of the charge distribution as

$$\oint_S \vec{D}_S \cdot d\vec{S} = \int_{vol} \rho_v \, dv$$

## 3.7 Applications of Gauss's Law

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface)*. The surface over which Gauss's law is applied must be closed, but it can be made up of several surface elements. Thus the defining conditions of a *special* **Gaussian *surface*** are

*a- The surface is closed.*

*b- At each point of the surface $\vec{D}$ is either normal or tangential to the surface, so that $\left(\vec{D}_S \cdot d\vec{S}\right)$ becomes either $\left(\vec{D}_S \, d\vec{S}\right)$ or $(zero)$, respectively*

*c- $\vec{D}$ is sectional constant over that part of the surface where $\vec{D}$ is normal.*

## 3.8 Symmetrical Charge Distributions:

### 3.8.1 Point Charge:

Suppose a point charge Q is located at the origin. To determine D at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a ***spherical surface*** centered at the origin is the ***Gaussian surface*** in this case and is shown in Figure 3.7.

$$\Psi = \oint_S \vec{D}_S \cdot d\vec{S} = D_r \oint d\vec{S}$$

$$\Psi = D_r \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \ d\theta \, d\emptyset = 4\pi \, r^2 \, D_r$$

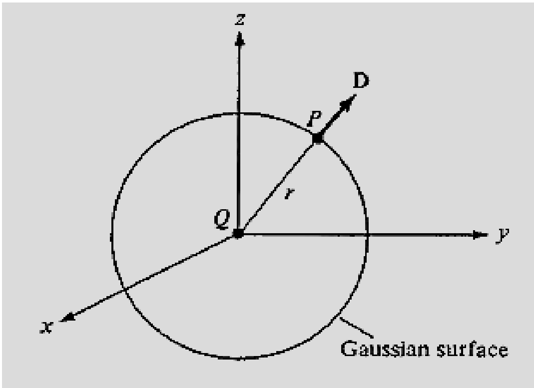



**Figure 3.7** Gaussian surface about a point charge

$$Q_{enclosed} = Q$$

$$\Psi = Q_{enclosed}$$

$$4\pi r^2 D_r = Q$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2} \, \boldsymbol{a}_r}$$

### 3.8.2 Infinite Line Charge

Suppose the infinite line of uniform charge $\rho_L\ C/m$ lies along the z-axis. To determine **D** at a point P, we choose a ***cylindrical surface*** containing P to satisfy symmetry condition as shown in Figure 3.8. **D** is constant on and normal to the cylindrical Gaussian surface; i.e., $= D_\rho \boldsymbol{a}_\rho$ . If we apply Gauss's law to an arbitrary length L of the line

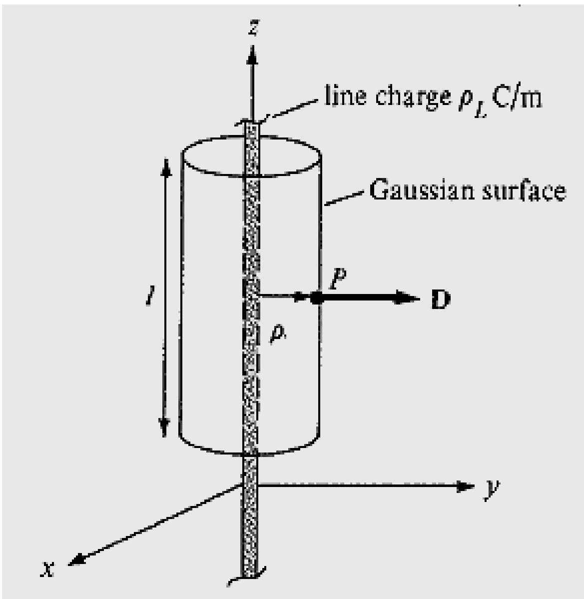


**Figure 3.8** Gaussian surface about an infinite line

$$\Psi = \int \vec{D}_S \cdot d\vec{S} = D_\rho \int d\vec{S}$$

$$\Psi = \vec{D}_\rho \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho\, d\emptyset\, dz$$

$$\Psi = 2\pi\, \rho\, l\, D_\rho$$

$$Q_{enclosed} = \int \rho_L\, dl$$

$$Q_{enclosed} = \rho_L \int_{z=0}^{l} dz = l\, \rho_L$$

$$\Psi = Q_{enclosed}$$

$$2\pi\, \rho\, l\, D_\rho = l\, \rho_L$$

$$D_\rho = \frac{\rho_L}{2\pi\, \rho}$$

$$\vec{D} = \frac{\rho_L}{2\pi\, \rho} \boldsymbol{a}_\rho$$

## 3.8.3 Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge $\rho_v\ C/m^3$. To determine D everywhere, we construct Gaussian surfaces for cases r < $a$, and r > $a$ separately. Since the charge has spherical symmetry, it is obvious that a ***spherical surface*** is an appropriate Gaussian surface.

For r < $a$, the total charge enclosed by the spherical surface of radius r, as shown in Figure 3.9 (a), is

$$\Psi = \oint \vec{D}_S \cdot d\vec{S} = \vec{D}_r \oint d\vec{S} = \vec{D}_r \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta d\emptyset = 4\pi r^2 \vec{D}_r$$

$$Q_{enclosed} = \int_{vol} \rho_v \, dv = \rho_v \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{r} r^2 \sin\theta \, d\theta d\emptyset \, dr = \frac{4}{3}\rho_v \pi r^3$$

$$\Psi = Q_{enclosed} \qquad (Gauss's\ Law)$$

$$4\pi r^2 \vec{D}_r = \frac{4}{3}\rho_v \pi r^3$$

$$\vec{D}_r = \frac{\rho_v}{3} r$$

$$\vec{D} = \frac{\rho_v}{3} r\, \boldsymbol{a}_r \qquad (r < a)$$

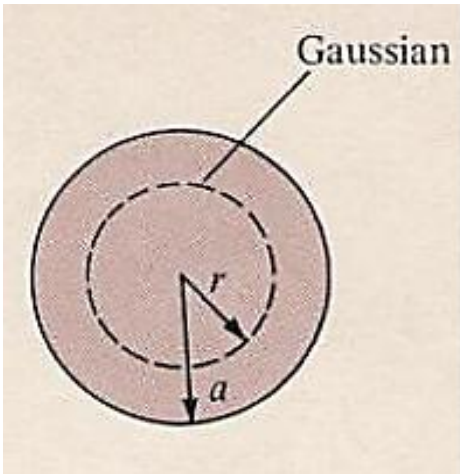



**Figure** 3.9 (a)

For r $\geq a$, the Gaussian surface is shown in Figure 3.9(b). The charge enclosed by the surface is the entire charge in this case, i.e.,

$$Q_{enclosed} = \int_{vol} \rho_v \, dv = \rho_v \int_{\emptyset=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{a} r^2 \sin\theta \, d\theta \, d\emptyset \, dr = \frac{4}{3} \rho_v \, \pi a^3$$

$$\Psi = Q_{enclosed} \qquad (Gauss's\ Law)$$

$$4\pi r^2 \vec{D}_r = \frac{4}{3} \rho_v \, \pi a^3$$

$$\vec{D}_r = \frac{a^3}{3\, r^2} \rho_v$$

$$\vec{D} = \frac{a^3}{3\, r^2} \rho_v \, \boldsymbol{a}_r \qquad (r \geq a)$$

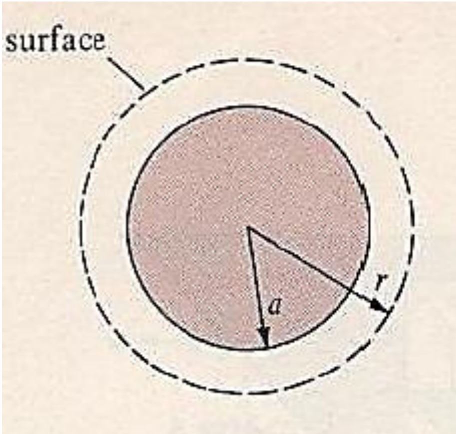
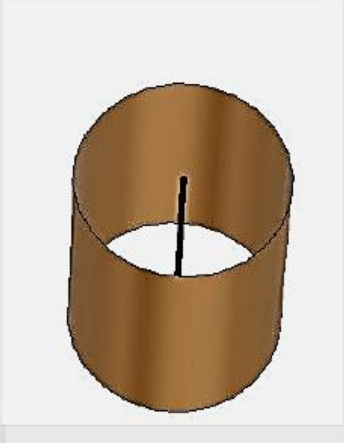



**Figure** 3.9 (b)

***Example:*** A uniform line charge of $\rho_L = 3\mu$ C/m lies along the z axis, and a concentric circular cylinder of radius 2 m has $\rho_s = \frac{-1.5}{4\pi}$ $\mu$C/m$^2$. Both distributions are infinite in extent with z. Use Gauss's law to find D in all regions?

***Solution:***



$1-$ *The region* $0 < \rho < 2$

*Using Gaussian surface cylinder* $\rho$

$$\Psi = Q_{enclosed}$$

$$\Psi = D_\rho \int_{z=0}^{l} \int_{\emptyset=0}^{2\pi} \rho d\emptyset dz = 2\pi\rho L D_\rho$$

$$Q_{enclosed} = \int \rho_L dL = \rho_L \int_{z=0}^{L} d_z = L\,\rho_L$$

$$2\pi\rho L D_\rho = L\,\rho_L$$

$$D = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho = \frac{3\times10^{-6}}{2\pi\rho}\mathbf{a}_\rho = \frac{0.477}{\rho}\mathbf{a}_\rho \quad \mu\,C/m^2$$

$2-$ *The region* $2 < \rho$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_1 = L\,\rho_L$$

$$Q_2 = \int \rho_s\, dS$$

$$Q_2 = \rho_s \int_0^L \int_0^{2\pi} \rho\, d\emptyset\, dz = 2\times2\pi\times L\times\rho_s = 4\pi L\rho_s$$

$$Q_{enclosed} = Q_1 + Q_2 = L\,\rho_L + 4\pi L\rho_s$$

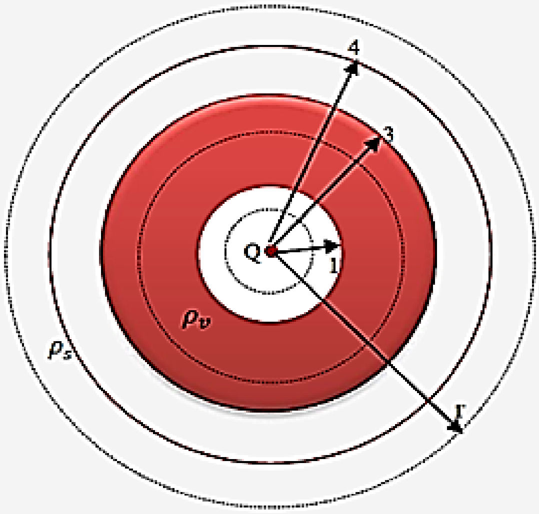
$$\Psi = Q_{enclosed}$$

$$2\pi\rho L D_\rho = L\,\rho_L + 4\pi L\rho_s$$

$$D = \frac{\rho_L + 4\pi\rho_s}{2\pi\rho}\,\mathbf{a}_\rho = \frac{0.239}{\rho}\mathbf{a}_\rho \quad \mu\,C/m^2$$

***Example:*** *A point charge $Q=2\pi$ c at the origin , a volume charge density of 4 $c/m^3$ at the region $1 \le r \le 3$ and a sheet of charge -6 $c/m^2$ has $r = 4$, Find D at $r = 0.5$ , $r = 2$ , $r = 5$?*

***Solution:***



*Gaussian surface*

*D at* $r = 0.5$

*D is due to a point charge*

$$D = \frac{Q}{4\pi r^2}\,\mathbf{a}_r = \frac{2\pi}{4\pi(0.5)^2}\,\mathbf{a}_r = 2\,\mathbf{a}_r$$

***D** at* $\mathbf{r} = \mathbf{2}$

$$\Psi = Q_{enclosed}$$

$$\Psi = 4\pi r^2 D_r$$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_2 = \int_{vol} \rho_v\, d_v = 4\int_0^{2\pi}\int_0^{\pi}\int_1^{2} r^2 \sin\theta\, dr d\theta d\emptyset = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^2 = \frac{112\pi}{3}$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{112\pi}{3}$$

$$D = \frac{39.33}{4(2)^2}\mathbf{a}_r = 2.45\,\mathbf{a}_r$$

***D** at* $\mathbf{r} = \mathbf{5}$

$$Q_{enclosed} = Q_1 + Q_2 + Q_3$$

$$Q_2 = 4 * 2 * 2\pi * \left[\frac{r^3}{3}\right]_1^3 = \frac{416\pi}{3}$$

$$Q_3 = \int \rho_s\, dS = -6\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta d\emptyset = -6 * (4)^2 * 2\pi * 2 = -384\pi$$

$$\therefore 4\pi r^2 D_r = 2\pi + \frac{416\pi}{3} - 384\pi = -243.33\pi$$

$$D_r = \frac{-243.33\pi}{4\pi r^2}\,\mathbf{a}_r = \frac{-243.33}{4(5)^2}\,\mathbf{a}_r = -2.433\,\mathbf{a}_r$$

## 3.9 Differential Volume Element

We are now going to apply the methods of Gauss's law to a slightly different type of problem, one which does not possess any symmetry at all. At first glance it might seem that our case is hopeless, for without symmetry a **simple Gaussian surface cannot be chosen** such that the normal component of D is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties, and that is to choose such a very small closed surface that D is *almost* constant over the surface, and the small change in D may be adequately represented by using the first two terms of the Taylor's-series expansion for D. The result will become more nearly correct as the volume enclosed by the Gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

$$Charge\ enclosed\ in\ volume\ \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times volume\ \Delta v$$

The expression is an approximation which becomes better as $\Delta \upsilon$ becomes smaller.

***Example***: Find an approximate value for the total charge enclosed in an incremental volume of $10^{-9}$ $m^3$ located at the origin, if $D = e^{-x} \sin y\, a_x - e^{-x} \cos y\, a_y + 2z a_z$ $C/m^2$.

***Solution***:

$$\frac{\partial D_x}{\partial x} = e^{-x} \sin y\ , \quad \frac{\partial D_y}{\partial y} = e^{-x} \sin y\ , \quad \frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately 2 .If $\Delta \upsilon$ is $10^{-9} m^3$, then we have enclosed about 2nC.

## 3.10 Divergence (div)

There are two main indicators of the manner in which a vector field changes from point to point throughout space. The first of these is divergence, which will be examined here. It is a scalar and bears a similarity to the derivative of a function. The second is curl.

When the divergence of a vector field is nonzero, that region is said to contain sources or sinks, sources when the divergence is positive, sinks when negative. In static electric fields there is a correspondence between positive divergence, sources, and positive electric charge Q. Electric flux Ψ by definition originates on positive charge. Thus, a region which contains positive charges contains the sources of Ψ. The divergence of the electric flux density D will be positive in this region. A similar correspondence exists between negative divergence, sinks, and negative electric charge

$$\text{Divergence of A=div A}= \lim_{\Delta v \to 0} \frac{\oint \mathrm{A}\ .dS}{\Delta v}$$

$$\text{div D}=\nabla.\text{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(Cartesian)}$$

$$\text{div D}=\nabla.\text{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial \emptyset} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div D}=\nabla.\text{D} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\emptyset}{\partial \emptyset} \qquad \text{(spherical)}$$

## 3.11 Maxwell's First Equation (Electrostatics)

We now wish to consolidate the gains of the last two sections and to provide an interpretation of the divergence operation as it relates to electric flux density. The expressions developed there may be written as

$$\text{div D} = \lim_{\Delta v \to 0} \frac{\oint D_S \cdot dS}{\Delta v}$$

$$\text{div D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div D} = \rho_v$$

$$\nabla . \text{D} = \rho_v$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that ***the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there***. This equation is called the ***point form** of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed,

***Example***: in the region $a \leq \rho \leq b$, $\vec{\mathbf{D}} = \rho_0 \left(\frac{\rho^2 - a^2}{2\rho}\right) \mathbf{a}_\rho$,

and for $\rho > b$ $\vec{\mathbf{D}} = \rho_0 \left(\frac{b^2 - a^2}{2\rho}\right) \mathbf{a}_\rho$

for $\rho < a$ $\vec{\mathbf{D}} = \mathbf{0}$, find $\rho_v$ in all three regions?

***Solution***:

$1 -$ *for the region* $a \leq \rho \leq b$

$$D_\rho = \rho_0 \left(\frac{\rho^2 - a^2}{2\rho}\right) \quad , \quad D_\emptyset = 0 \quad , \quad D_z = 0$$

$$\nabla . \text{D} = \rho_v$$

$$\rho_v = \nabla . \text{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho D_\rho + \frac{1}{\rho} \frac{\partial D_\emptyset}{\partial \emptyset} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \left(\rho_0 \left(\frac{\rho^2 - a^2}{2\rho}\right)\right) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho_0 \left(\frac{\rho^2 - a^2}{2}\right) = \frac{\rho_0}{2\rho} \frac{\partial}{\partial \rho} (\rho^2 - a^2) = \frac{\rho_0}{2\rho} \times 2\rho = \rho_0 \; C/m^3$$

$2 - \ for\ the\ region\ \ \rho > b$

$$\rho_v = \nabla.\mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\rho_0\left(\frac{b^2 - a^2}{2\rho}\right)\right) == \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho_0\left(\frac{b^2 - a^2}{2}\right) = \frac{\rho_0}{2\rho}\frac{\partial}{\partial\rho}(b^2 - a^2) = 0$$

$3 - \ for\ the\ region\ \ \rho < a$

$$\rho_v = \nabla.\mathrm{D} = \rho_v = \nabla.0 = 0$$

***Example***: Let $\vec{\mathbf{D}} = 5r^2\mathbf{a}_r\ \mathrm{mC/m^2}$, for $r \le 0.08$ m and $\vec{\mathbf{D}} = \frac{0.1}{r^2}\mathbf{a}_r\ \mathrm{mC/m^2}$, for $r > 0.08$ m, (a) find $\rho_v$ at r =0.06m, (b) find $\rho_v$ at r =0.1m, (c) what surface charge density could be located at r =0.08m to caused D=0 for $> 0.08$ ?

***Solution***:

$(\boldsymbol{a})\ \ \rho_v\ at\ r = 0.06$

$for\ r < 0.08 \qquad \vec{\mathbf{D}} = 5r^2\mathbf{a}_r\ \mathrm{mC/m^2}$

$$\rho_v = \nabla.D = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\emptyset}{\partial\emptyset}$$

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}r^2(5r^2) = \frac{1}{r^2}\frac{\partial}{\partial r}5r^4 = \frac{1}{r^2}\ 20r^3 = 20r\ \ mC/m^3$$

$$\rho_{v\,at\ r=0.06} = 20(0.06)\times 10^{-3} = 1.2\ mC/m^3$$

$(\boldsymbol{b})\ \ \rho_v\ at\ r = 0.1$

$for\ r > 0.08$

$$\vec{\mathbf{D}} = \frac{0.1}{r^2}\mathbf{a}_r\ \ \mathrm{mC/m^2}$$

$$\rho_v = \nabla.D = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{0.1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}0.1 = 0$$

$(\boldsymbol{c})\ \ D = 0 \Rightarrow \Psi = 0\ \ \Rightarrow Q_{enclosed} = 0$

$$Q_{enclosed} = Q_1 + Q_2$$

$$Q_2 = -Q_1 = -\int \rho_v dv = -\int_0^{2\pi}\int_0^{\pi}\int_0^{0.08} 20r\, r^2 \sin\theta \, dr\, d\theta d\emptyset$$

$$Q_2 = -\int_0^{2\pi}\int_0^{\pi}\int_0^{0.08} 20\, r^3 \sin\theta \, dr\, d\theta d\emptyset = -\,20 \times 2\pi \times 2 \times \left[\frac{r^4}{4}\right]_0^{0.08} = -2.57\,\mu C$$

$$Q_2 = \int \rho_s dS = \rho_s \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \;\, d\theta d\emptyset = (0.08)^2 \times 2 \times 2\pi \times \rho_s$$

$$-2.57 \times 10^{-6} = 0.0804\, \rho_s$$

$$\rho_s = -32\,\mu C/m^2$$

## 3.12 The Divergence Theorem

Gauss' law states that the closed surface integral of **D**. $dS$ is equal to the charge enclosed. If the charge density function $\rho_v$ is known throughout the volume, then the charge enclosed may be obtained from an integration of $\rho_v$ throughout the volume. Thus

$$\oint D_S \;.\, dS = \int_{vol} \rho_v dv$$

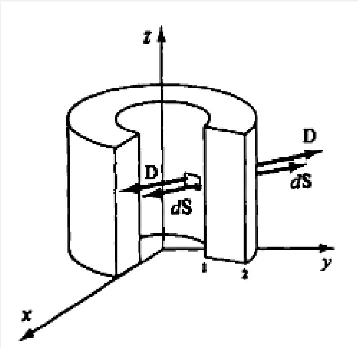
But $\nabla.\mathrm{D} = \rho_v$ and so

$$\boxed{\oint D_S \;.\, dS = \int_{vol} (\nabla.\mathrm{D}) dv}$$

This is the ***divergence theorem***, also known as *Gauss' divergence theorem.* Of course, the volume v is that which is enclosed by the surface S.

***Example***: Given that $D = (10\rho^3/4)\ a\rho$, (C/m$^2$) in cylindrical coordinates, evaluate both sides of the divergence theorem for the volume enclosed by $\rho$= 1m, $\rho$= 2m, z = 0 and z = 10

***Solution***:




***The left side of divergence theorem is:***

$$\oint D_S\ .dS = \int_0^{10}\int_0^{2\pi}\frac{10\rho^3}{4}\rho d\emptyset dz = \frac{10}{4}\times\rho^4\times 2\pi\times 10$$

$$= 50\pi\ \rho^4 = 50\pi(2^4 - 1^4) = 750\pi$$

***The right side is:***

$$\int_{vol}(\nabla.\mathrm{D})dv = \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho\right)dv = \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{10\rho^3}{4}\right)\right)dv$$

$$= \int_{vol}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{10\rho^4}{4}\right)\right)dv = \int_{vol}(10\rho^2)dv = \int_0^{10}\int_0^{2\pi}\int_1^2 10\rho^2\ \rho\ d\rho\ d\emptyset\ dz$$

$$= 10\times 2\pi\times 10\times\left[\frac{\rho^4}{4}\right]_1^2 = 750\pi$$

***Example***: Given that $\vec{\mathbf{D}} = 10\sin\theta\,\mathbf{a}_r + 2\cos\theta\,\mathbf{a}_\theta$, evaluate both sides of the divergence theorem for the volume enclosed by the shell r = 2?

***Solution***:

***The left side of divergence theorem is:***

$$\oint D_S\ .dS = \oint(10\sin\theta\,\mathbf{a}_r + 2\cos\theta\,\mathbf{a}_\theta).r^2\sin\theta\ d\theta d\emptyset\,\mathbf{a}_r$$

$$\oint D_S\ .dS = \int_0^{2\pi}\int_0^{\pi}10r^2\sin^2\theta\ d\theta d\emptyset = 10(2)^2\times 2\pi\times\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 80\pi\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi} = 80\pi\left[\frac{1}{2}\pi\right] = 40\pi^2$$

***The right side is:***

$$\int_{vol}(\nabla.\mathrm{D})dv = \int_{vol}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2 D_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,D_\theta)\right)dv$$

$$= \int_{vol} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 (10 \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \, 2\cos\theta) \right) dv$$

$$= \int_{vol} \left( \frac{1}{r^2} \frac{\partial}{\partial r} 10 r^2 \sin\theta + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \sin 2\theta \right) dv$$

$$= \int_{vol} \left( \frac{20}{r} \sin\theta + \frac{2}{r \sin\theta} \cos 2\theta \right) dv$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} \left( \frac{20}{r} \sin\theta + \frac{2}{r \sin\theta} \cos 2\theta \right) r^2 \sin\theta \;\, dr d\theta d\emptyset$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 20 r \sin^2\theta \;\, dr d\theta d\emptyset + \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 2r \cos 2\theta \;\, dr d\theta d\emptyset$$

$$= [10r^2]_0^2 \left[ \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi} [\emptyset]_0^{2\pi} + \; [r^2]_0^2 \left[ \frac{1}{2} \sin 2\theta \right]_0^{\pi} [\emptyset]_0^{2\pi}$$

$$[10 \times 4] \left[ \frac{1}{2}\pi \right] [2\pi] = 40\pi^2$$

## 3.9 Electrical Potential

Suppose we wish to move a point charge $Q$ from point $A$ to point $B$ in an electric field $\boldsymbol{E}$ as shown in Figure 3.10. From Coulomb's law, the force on $Q$ is $F = QE$ so that the *work done* in displacing the charge by $dl$ is

$$dw = -\vec{F} \cdot d\vec{L} = -\,Q\vec{E} \cdot d\vec{L}$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving $Q$ from $A$ to $B$ is

$$w = -\,Q \int_{initial}^{final} \vec{E} \cdot d\vec{L}$$

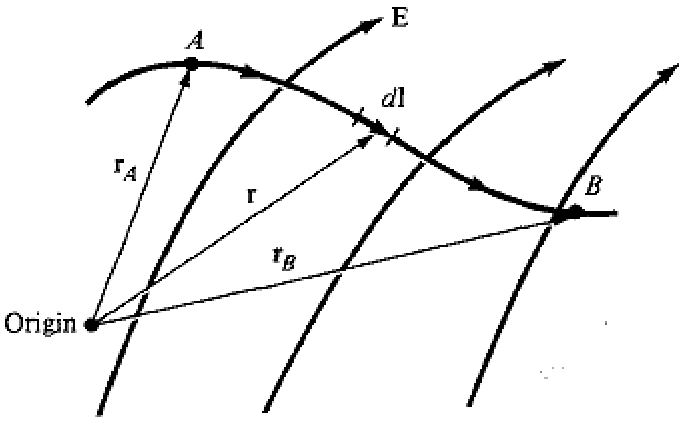



**Figure 3.10** Displacement of point charge $Q$ in an electrostatic field $\boldsymbol{E}$.

***Example:*** An electrostatic field is given by E = ($x$/2 + 2y) $\mathbf{a_x}$ + $2x$ $\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q$ = -20 $\mu C$ *(a)* from the origin to (4, 0, 0) m, and *(b)* from (4, 0, 0) m to (4, 2, 0) m?

***Solution:***

**(a)** The first path is along x-axis, so that the dL = $dx\mathbf{a_x}$

$$W = -Q\int_{inital}^{final} E.dL = -(-20\times10^{-6})\int_0^4\left(\frac{x}{2}+2y\right).dx = 20\times10^{-6}\left[\frac{x^2}{4}\right]_0^4 = 80\mu\,\mathrm{J}$$

**(b)** The second path is in the $a_y$ direction, so that the dL = $dy\mathbf{a_y}$

$$W = -(-20\times10^{-6})\int_0^2 2x.dy = 20\times10^{-6}\times2\times4[y]_0^2 = 160\times10^{-6}\times2 = 320\mu\,\mathrm{J}$$

***Example:*** Find the work done in moving a point charge Q = 5 μC from the origin to (2 m, π/4, π/2) spherical coordinates, in the field $E = 5e^{-r/4}\mathrm{a}_r + \frac{10}{r\sin\theta}\mathrm{a}_\phi \ (V/m)$?

***Solution:***

$$\mathrm{dL} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\ \sin\theta\ d\phi\ \mathbf{a}_\phi$$

$$W = -Q\int_{inital}^{final} E.dL$$

$$= -5\mu\int_{inital}^{final}\left(5e^{-r/4}\mathrm{a}_r + \frac{10}{r\sin\theta}\mathrm{a}_\phi\ \right).(dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\ \sin\theta\ d\phi\ \mathbf{a}_\phi)$$

$$= -5\mu\int_0^2 5e^{-r/4}dr - 5\mu\int_0^{\pi/2}\frac{10}{r\sin\theta}r\ \sin\theta\ d\phi$$

$$= -25\mu(-4)|e^{-r/4}|_0^2 - 5\mu(10)|\phi|_0^{\pi/2}$$

$$= 100\mu\left(e^{-1/2} - e^0\right)\ - 50\mu\times\frac{\pi}{2} = -117.9\mu\,\mathrm{J}$$

***Example:*** uniform line charge lie along z-axis, determine the work expended in carrying Q from a to b along:

**(a)** Circular path?

***(b)*** Radial path?

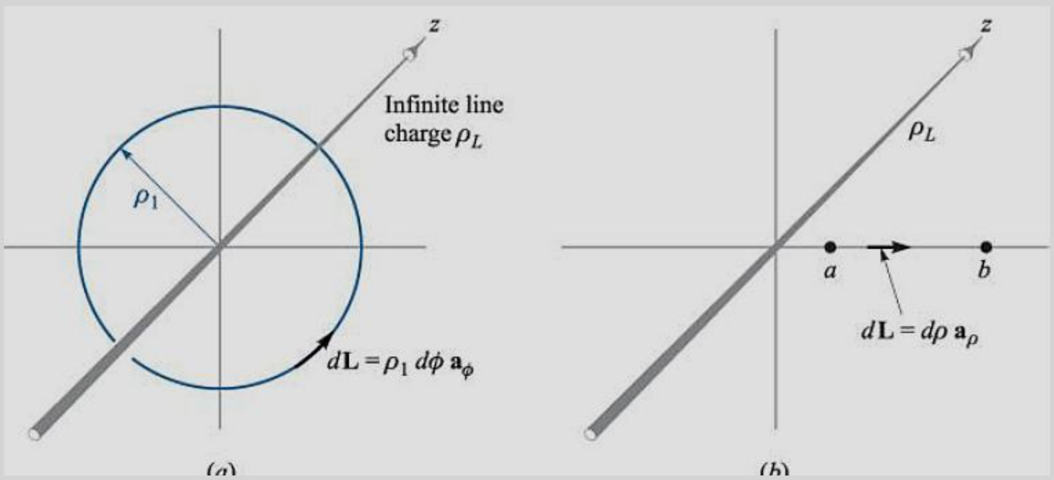



Figure: (a) A circular path and (b) a radial path along which a charge of Q is carried in the field of an infinite line charge.

***Solution***:

The electric field of a line charge is $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$

**(a)** for circular path $dL = \rho d\emptyset \mathbf{a}_\emptyset$

$$W = -Q\int_B^A \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho \,.\, \rho d\emptyset \mathbf{a}_\emptyset = 0 \qquad (\mathrm{a}_\rho . \mathrm{a}_\emptyset) = 0$$

**(b)** for radial path $dL = d\rho \mathbf{a}_\rho$

$$W = -Q\int_{inital}^{final} \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho \,.\, d\rho \mathbf{a}_\rho = -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon_o}\frac{d\rho}{\rho} = -Q\frac{\rho_L}{2\pi\varepsilon_o}\int_b^a \frac{d\rho}{\rho}$$

$$W = \frac{-Q\rho_L}{2\pi\varepsilon_o}\ln\frac{b}{a}$$

## 3.10 Definition of Potential Difference and Potential

We are now ready to define a new concept from the expression for the work done by an external source in moving a charge $Q$ from one point to another in an electric field $\vec{E}$,

$$w = -Q\int_{initial}^{final} \vec{E} \cdot d\vec{L}$$

In much the same way as we defined the electric field intensity as the force on a *unit* test charge, we now define ***potential difference*** $V$ as the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$Potential\ Difference = V = -\int_{initial}^{final} \vec{E} \cdot d\vec{L}$$

Potential difference is measured in joules per coulomb, for which the *volt* is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_{B}^{A} \vec{E} \cdot d\vec{L}$$

$V_{AB}$ is positive if work is done in carrying the positive charge from $B$ to $A$.

## 3.11 The Potential Field of a Point Charge

The potential difference between points located at $r = r_A$ and $r = r_B$ in the field of a point charge $Q$ placed at the origin

$$\vec{\mathrm{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\boldsymbol{a}_r$$

$$d\vec{L} = dr\ \boldsymbol{a}_r$$

$$V_{AB} = -\int_{B}^{A} \vec{E} \cdot d\vec{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\boldsymbol{a}_r \cdot dr\ \boldsymbol{a}_r = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

or

$$V_{AB} = V_A - V_B$$

The potential difference between two points in the field of a point charge depends only on the distance of each point from the charge and does not depend on the particular path used to carry our unit charge from one point to the other

How might we conveniently define a zero reference for potential? The simplest possibility is to let $V = 0$ at infinity. If we let the point at $r = r_B$ recede to infinity the potential at $r_A$ becomes

$$V_A = \frac{Q}{4\pi\varepsilon_0 r_A} \qquad \left(\frac{1}{r_B} = \frac{1}{\infty} = 0\right)$$

Since there is no reason to identify this point with the A subscript,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

This expression defines the potential at any point distant r from a point charge $Q$ at the origin, the potential at infinite radius being taken as the zero reference.

A convenient method to express the potential without selecting a specific zero reference entails identifying $r_A$ as $r$ once again and letting $\frac{Q}{4\pi\varepsilon_0 r_A}$ be a constant. Then

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C_1$$

$C_1$ may be selected so that $V = 0$ at any desired value of $r$. We could also select the zero reference indirectly by electing to let $V$ be $V_0$ at $r = r_0$.

The **potential** at any point is the potential difference between that point and a chosen point in which the **potential** is **zero**.

***Example:*** Find the potential at $r_A$ = 5 m with respect to $r_B$ =*15* m due to a point charge *Q* =500 pC at the origin and zero reference at infinity?

***Solution:***

$$V_{AB} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{rA} - \frac{1}{rB}\right)$$

$$V_{AB} = \frac{500 \times 10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{5} - \frac{1}{15}\right) = 0.6\,V$$

The zero reference at infinite must be used to find $V_5$ and $V_{15}$

$$V_5 = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r5}\right) = \frac{500 \times 10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{5}\right) = 0.9\,V$$

$$V_{15} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r15}\right) = \frac{500 \times 10^{-12}}{4\pi\varepsilon_o}\left(\frac{1}{15}\right) = 0.3\,V$$

***Example:*** A 15-nC point charge is at the origin in free space. Calculate $V_1$ if point $P_1$ is located at $P_1$ (-2, 3, -1) and: *(a) V* = 0 at (6, 5, 4); (b) V =0 at infinity; (c) V = 5 V at (2, 0, 4)?

***Solution:***

***a-***

$$V_{p1} = \frac{Q}{4\pi\varepsilon_o R_{p1}} + c$$

$$C = V_{ref} - \frac{Q}{4\pi\varepsilon_o R_{ref}}$$

$$C = 0 - \frac{15 \times 10^{-9}}{4\pi\varepsilon_o\sqrt{6^2 + 5^2 + 4^2}} = -15.37$$

$$\therefore V_{p1} = \frac{15 \times 10^{-9}}{4\pi\varepsilon_o\sqrt{2^2 + 3^2 + 1^2}} - 15.37$$

$$V_{p1} = 20.7\,V$$

***b***- *V=0 at infinity*

$$V_{p1}=\frac{Q}{4\pi\varepsilon_o R_{p1}}=\frac{15\times10^{-9}}{4\pi\varepsilon_o\sqrt{2^2+3^2+1^2}}=36.04\,V$$

***c***-

$$C=V_{ref}-\frac{Q}{4\pi\varepsilon_o R_{ref}}=5-\frac{15\times10^{-9}}{4\pi\varepsilon_o\sqrt{2^2+0^2+4^2}}=-25.15$$

$$V_{p1}=36.04-25.15=10.89\,V$$

## 3.12 The Potential Field of a Line Charge

The potential difference between points located at $\rho=\ a$ and $\rho=\ b$ in the field of a point charge $Q$ placed at the origin

$$\vec{E}=\frac{\rho_L}{2\pi\varepsilon_0\rho}\boldsymbol{a}_\rho$$

and $d\vec{L}=d\rho\boldsymbol{a}_\rho$

$$V_{AB}=-\int_B^A\vec{E}\cdot d\vec{L}=-\int_b^a\frac{\rho_L}{2\pi\varepsilon_0\rho}\boldsymbol{a}_\rho\cdot d\rho\boldsymbol{a}_\rho=-\int_b^a\frac{\rho_L}{2\pi\varepsilon_0\rho}d\rho$$

$$V_{AB}=\frac{\rho_L}{2\pi\varepsilon_0}ln\frac{b}{a}$$

***Example:*** For a line charge $\rho_L=10^{-9}/2$ C/m on the z-axis, find $V_{AB}$, where $A$ is (2m, π/2, 0) and $B$ is (4m, π, 5m) ?

***Solution:***

$$V_{AB}=\frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{b}{a}$$

$$V_{AB}=\frac{0.5\times10^{-9}}{2\pi\varepsilon_o}\ln\frac{4}{2}=6.24\,V$$

***Example*:** A point charge 5 $nC$ is located at $(-3, 4, 0)$ while line $y = 1, z = 1$ carries uniform charge2 $nC/m$.

(a) If $V = 0\ V$ at $O(0,0,0)$, find $V$ at A$(5, 0, 1)$.

(b) If $V = 100\ V$ at $B(1, 2, 1)$, find $V$at $C(-2, 5, 3)$.

(c) If $V = -5\ V$ at $O$, find $V_{BC}$.

***Solution:***

(a) $V = V_Q + V_L$

$$V_{OA} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_O} - \frac{1}{r_A}\right] + \frac{\rho_L}{2\pi\varepsilon_0} ln\frac{\rho_A}{\rho_O}$$

$$\rho_O = (0,0,0) - (0,1,1) = \sqrt{2}$$

$$r_O = (0,0,0) - (-3, 4, 0) = 5$$

$$\rho_A = (5,0,1) - (5,1,1) = 1$$

$$r_A = (5,0,1) - (-3, 4, 0) = 9$$

$$V_O - V_A = \frac{5 * 10^{-9}}{4\pi * \frac{10^{-9}}{36\pi}}\left[\frac{1}{5} - \frac{1}{9}\right] + \frac{2 * 10^{-9}}{2\pi * \frac{10^{-9}}{36\pi}}\ ln\ \frac{1}{\sqrt{2}}$$

$$0 - V_A = 45\left[\frac{1}{5} - \frac{1}{9}\right] + 36\ ln\ \frac{1}{\sqrt{2}}$$

$$-V_A = 4 + 36\ ln\ \frac{1}{\sqrt{2}}$$

$$\therefore\ V_A = 8.447\ V$$

(b) $\rho_B = (1,2,1) - (1,1,1) = 1$

$$r_B = (1,2,1) - (-3,4,0) = \sqrt{21}$$

$$\rho_C = (-2,5,3) - (-2,1,1) = \sqrt{20}$$

$$r_C = (-2,5,3) - (-3,4,0) = \sqrt{11}$$

$$V_C - V_B = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_C} - \frac{1}{r_B}\right] + \frac{\rho_L}{2\pi\varepsilon_0} ln \frac{\rho_B}{\rho_C}$$

$$V_C - V_B = \frac{5*10^{-9}}{4\pi * \frac{10^{-9}}{36\pi}}\left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right] + \frac{2*10^{-9}}{2\pi * \frac{10^{-9}}{36\pi}}\ ln\ \frac{1}{\sqrt{20}}$$

$$V_C - 100 = 45\left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right] + 36\ ln\ \frac{1}{\sqrt{20}}$$

$$V_C = -50.175\ V$$

$or$ $\qquad V_C = 49.825\,V$

(c) $V_{BC} = V_C - V_B = 49.825 - 100$

$$= -50.175\,V$$

## 3.13 The Potential Field of a System of Charges

The potential field of a single point charge, which we shall identify as $Q_1$ and locate at $r_1$, involves only the distance $|r - r_1|$ from $Q_1$ to the point at $r$. For a zero reference at infinity, we have

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r} - \vec{r}_1|}$$

The potential due to two charges, $Q_1$ at $r_1$ and $Q_2$ at $r_2$, is a function only of $|r - r_1|$ and $|r - r_2|$the distances from $Q_1$ and $Q_2$ to the field point, respectively.

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\vec{r} - \vec{r}_2|}$$

If the charge distribution takes the form of a line charge, a surface charge, a volume charge the integration is along the line or over the surface or volume:

$$V = \int \frac{\rho_L \, dL}{4\pi \, \varepsilon_0 \, |\vec{R}|}$$

$$V = \int_S \frac{\rho_S \, dS}{4\pi \, \varepsilon_0 \, |\vec{R}|}$$

$$V = \int_{vol} \frac{\rho_v \, dV}{4\pi \, \varepsilon_0 \, |\vec{R}|}$$

***Example:*** Five equal point charges, Q=2 nC, are located at x=2, 3, 4, 5, 6 m. Find the potential at the origin?

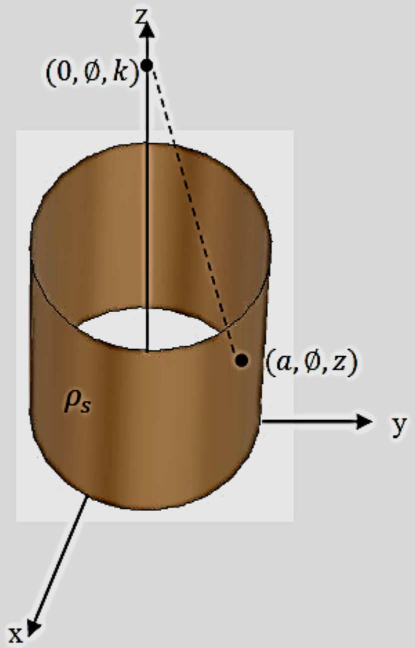
***Solution:***

$$V = V_1 + V_2 + V_3 + V_4 + V_5$$

$$V_1 = \frac{Q}{4\pi\varepsilon_o r_1} \quad , \quad V_2 = \frac{Q}{4\pi\varepsilon_o r_2}$$

$$\therefore\ V = \frac{2\times10^{-9}}{4\pi\varepsilon_o}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right) = 261\,V$$

***Example:*** Find the potential V at (0, 0, K) for cylindrical surface charge $\rho_s = \rho_{s0}$, $0 \le z \le h$ and $\rho = a$?




***Solution:***

$$V = \int_s \frac{\rho_s ds}{4\pi\varepsilon_o |R|}$$

$$ds = \rho\, d\emptyset\, dz \ = \ a\, d\emptyset\, dz$$

$$R = -a\,\mathrm{a}_\rho + (k-z)\mathrm{a}_z$$

$$|R| = \sqrt{a^2 + (k-z)^2}$$

$$V = \int_0^h \int_0^{2\pi} \frac{\rho_{so} a\, d\emptyset\, dz}{4\pi\varepsilon_o\sqrt{a^2+(k-z)^2}}$$

$$V = \frac{\rho_{so} a}{4\pi\varepsilon_o} \times 2\pi \int_0^h \frac{dz}{\sqrt{a^2+(k-z)^2}}$$

$$V = \frac{-a\rho_{so}}{2\varepsilon_o}\int_0^h \frac{-\,dz}{a\sqrt{1+\left(\frac{k-z}{a}\right)^2}}$$

$$V = \frac{-\rho_{so}}{2\varepsilon_o}\left[\sinh^{-1}\left(\frac{k-z}{a}\right)\right]_0^h$$

$$V = \frac{\rho_{so}}{2\varepsilon_o}\left[\sinh^{-1}\left(\frac{k-h}{a}\right) - \sinh^{-1}\left(\frac{k}{a}\right)\right]$$

$$*\ \sinh^{-1} u = \ln\left(u + \sqrt{u^2+1}\right)$$

$$V=\frac{\rho_{so}}{2\varepsilon_o}\left[\ln\left(\left(\frac{k-h}{a}\right)+\sqrt{\left(\frac{k-h}{a}\right)^2+1}\right)-\ln\left(\left(\frac{k}{a}\right)+\sqrt{\left(\frac{k}{a}\right)^2+1}\right)\right]$$

$$\therefore\ V=\frac{\rho_{so}}{2\varepsilon_o}\ln\left(\frac{k+\sqrt{a^2+k^2}}{k-h+\sqrt{a^2+(k-h)^2}}\right)$$

## 3.16 Gradient

The vector field ∇V (also written grad **V)** is called the *gradient* of the scalar function **V**

$$\nabla V=\frac{\partial V}{\partial x}\mathbf{a}_x+\frac{\partial V}{\partial y}\mathbf{a}_y+\frac{\partial V}{\partial z}\mathbf{a}_z \qquad (\text{Cartesian})$$

$$\nabla V=\frac{\partial V}{\partial \rho}\mathbf{a}_\rho+\frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\mathbf{a}_\emptyset+\frac{\partial V}{\partial z}\mathbf{a}_z \qquad (\text{cylindrical})$$

$$\nabla V=\frac{\partial V}{\partial r}\mathbf{a}_r+\frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta+\frac{1}{r\sin\theta}\frac{\partial V}{\partial \emptyset}\mathbf{a}_\emptyset \qquad (\text{spherical})$$

## 3.17 Relationship Between $\vec{E}$ and $V$

The electric field intensity E may be obtained when the potential function **V** is known by simply taking the negative of the gradient of **V**. The gradient was found to be a vector normal to the equipotential surfaces, directed to a positive change in **V**. With the negative sign here, the $\vec{E}$ field is found to be directed from higher to lower levels of potential **V**

$$\vec{E}=-\nabla V$$

***Example:*** Given the potential field, $V = 2x^2 y - 5z$, and a point $P(-4, 3, 6)$, find at point $P$: the potential $V$, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density $\rho_v$?

***Solution:***

The potential at P is:

$$V = 2(-4)^2(3) - 5(6) = 66\,V$$

The electric field intensity E is :

$$E = -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

$$E = \frac{-\partial}{\partial x}(2x2\,y - 5z)\mathbf{a}_x + \frac{-\partial}{\partial y}(2x2\,y - 5z)\mathbf{a}_y + \frac{-\partial}{\partial z}(2x2\,y - 5z)\mathbf{a}_z$$

$$E = -4xy\,\mathbf{a}_x - 2x^2\,\mathbf{a}_y + 5\mathbf{a}_z$$

E at the point P is:

$$E = 48\,\mathbf{a}_x - 32\,\mathbf{a}_y + 5\mathbf{a}_z$$

$$D = \varepsilon_o E = \varepsilon_o\left(-4xy\,\mathbf{a}_x - 2x^2\,\mathbf{a}_y + 5\mathbf{a}_z\right)$$

$$\rho_v = \nabla . D = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} = -4\varepsilon_o\,y$$

$$\rho_{v\,(at\,point\,P)} = -4\varepsilon_o(3) = -106.2\,PC/m^3$$

***Example:*** Given the potential field in cylindrical coordinates $V = \frac{100}{z^2+1}\rho\cos\emptyset$ , and point $P$ $\rho = 3$ m, $\emptyset = 60^o$ , $z$=2 m, in free space find at point $P$: the potential $V$, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density $\rho_v$?

***Solution:***

The potential field V at the point P is:

$$V = \frac{100}{2^2+1}(3)\cos(60) = 30\,V$$

$$E = -\nabla V = -\left(\frac{\partial V}{\partial\rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial\emptyset}\mathbf{a}_\emptyset + \frac{\partial V}{\partial z}\,\mathbf{a}_z\right)$$

$$E = \left(\frac{\partial}{\partial\rho}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\emptyset}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\mathbf{a}_\emptyset + \frac{\partial}{\partial z}\left(\frac{100}{z^2+1}\,\rho\cos\emptyset\right)\,\mathbf{a}_z\right)$$

$$E = \frac{-100}{z^2+1}\cos\emptyset\;\mathbf{a}_\rho + \frac{-100}{z^2+1}\sin\emptyset\;\mathbf{a}_\emptyset + \frac{-200\,z}{(z^2+1)^2}\,\rho\cos\emptyset\;\mathbf{a}_z$$

$$E\ at\ p\ = \frac{-100}{2^2+1}\cos 60\;\mathbf{a}_\rho + \frac{-100}{2^2+1}\sin 60\;\mathbf{a}_\emptyset + \frac{-200(2)}{(2^2+1)^2}\,3\cos 60\;\mathbf{a}_z$$

$$E\ at\ P\ = -10\,\mathbf{a}_\rho - 17.3\,\mathbf{a}_\emptyset + 24\,\mathbf{a}_z$$

$$D = \varepsilon_o E = \varepsilon_o\left(\frac{-100}{z^2+1}\cos\emptyset\;\mathbf{a}_\rho + \frac{-100}{z^2+1}\sin\emptyset\;\mathbf{a}_\emptyset + \frac{-200\,z}{(z^2+1)^2}\,\rho\cos\emptyset\;\mathbf{a}_z\right)$$

$$\rho_v = \nabla.D = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho D_\rho + \frac{1}{\rho}\frac{\partial D_\emptyset}{\partial\emptyset} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \frac{1}{\rho}\frac{\partial}{\partial\emptyset}\left(\frac{-100}{z^2+1}\sin\emptyset\right) + \frac{\partial}{\partial z}\left(\frac{-200\,z}{(z^2+1)^2}\,\rho\cos\emptyset\right)$$

$$\rho_v = \frac{1}{\rho}\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \frac{1}{\rho}\left(\frac{-100}{z^2+1}\cos\emptyset\right) + \left(\frac{(z^2+1)^2\times(-200) - 200\,z\times 4z(z^2+1)}{(z^2+1)^4}\,\rho\cos\emptyset\right)$$

## 3.18 CURL OF A VECTOR

The curl of $\vec{A}$ is a rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \left( \lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{L}}{\Delta S} \right) \boldsymbol{a}_{n_{max}}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

**or**

$$\vec{\nabla} \times \vec{A} = \left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \boldsymbol{a}_x + \left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \boldsymbol{a}_y + \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \boldsymbol{a}_z \quad \boldsymbol{Cartesian}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \boldsymbol{a}_\rho & \rho \boldsymbol{a}_\emptyset & \boldsymbol{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\emptyset & A_z \end{vmatrix}$$

**or**

$$\vec{\nabla} \times \vec{A} = \left[ \frac{1}{\rho} \frac{\partial A_z}{\partial \emptyset} - \frac{\partial A_\emptyset}{\partial z} \right] \boldsymbol{a}_\rho + \left[ \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \boldsymbol{a}_\emptyset + \frac{1}{\boldsymbol{\rho}} \left[ \frac{\partial (\rho A_\emptyset)}{\partial \rho} - \frac{\partial A_\rho}{\partial \emptyset} \right] \boldsymbol{a}_z \quad \boldsymbol{Cylindrical}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \ \sin\theta} \begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\ sin\theta\ \boldsymbol{a}_\emptyset \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_r & rA_\theta & r\ sin\theta\ A_\emptyset \end{vmatrix}$$

or

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\emptyset \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\emptyset}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\emptyset} - \frac{\partial(rA_\emptyset)}{\partial r}\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{\boldsymbol{r}}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\boldsymbol{a}_\emptyset \qquad \boldsymbol{spherical}$$

Frequently useful are two properties of the curl operator:

*1) If the divergence of a curl is zero;* $(\nabla \times \vec{A}) = 0$, *then the vector field* $\vec{A}$ *is the Electric Field.*

*2) The curl of a gradient is the zero vector;* $(\nabla \times (\nabla\vec{A}) = 0$

***Example:*** For a vector field $\underline{\vec{A}}$, show explicitly that $\nabla \bullet \nabla \times \vec{A} = 0$; that is the divergence of the curl of any vector field is zero.

***Solution:***

For simplicity, assume that $\vec{A}$ is in Cartesian coordinates.

$$\nabla \bullet \nabla \times \vec{A} = \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right) \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$$= \mathbf{0}$$

# *Home work*

$\boldsymbol{Q_1}$ Given that $\vec{D} = z\,\rho\,cos^2\emptyset\,\hat{a}_z \;\frac{C}{m^2}$ calculate the charge density at $\left(1, \frac{\pi}{4}, 3\right)$ and the total charge enclosed by the cylinder of radius 1 $m$ with $-2 \le z \le 2\ m$. ***Answer*** $\rho_v = 0.5\ C/m^3$ $Q = \frac{4\pi}{3}\ C$ SD (134)

$\boldsymbol{Q_2}$ Find the volume charge density in a field $\vec{D} = (y + e^x)\hat{a}_x - ye^{-y}\hat{a}_y + z\hat{a}_z \;\frac{C}{m^2}$

$\boldsymbol{Q_3}$ The flux density $\vec{D} = \frac{r}{3}\,\hat{a}_r\ nC/m^2$ is in free space:

a) Find $\vec{E}\ at\ r = 0.2\ m.$
b) Find the total electric flux leaving the sphere of $r = 0.2\ m.$
c) Find the total charge with the sphere of $r = 0.3\ m.$

***Answer*** a) $7.5295\ \hat{a}_r\ V/m$ b) $Q = 33.51\ PC$ c) $Q = 113.097\ PC$

$\boldsymbol{Q_4}$ If a sphere of radius $10\ cm$ has a charge density $\rho_v = 15r^3\ C/m^3$, then determine $\vec{D}$ at $r \le 10\ cm$ and $r \ge 10\ cm.$

***Answer***

$$\vec{D} = \frac{10\ r^4}{4}\ V/m$$

$$\vec{D} = \frac{10^5}{4r^2}\ V/m$$

$\boldsymbol{Q_5}$ A charge distribution in free space has $\rho_v = 2\ r\ nC/m^3$ for $0 \le r \le 10\ cm$ and zero otherwise, determine $\vec{E}$ at $r = 2\ m$ and $r = 12\ m.$

***Answer*** $\vec{E} = \frac{2\times10^{-9}}{\varepsilon_0}\ V/m$ , $\vec{E} = \frac{10^{-5}}{288\ \varepsilon_0}\ V/m$

$Q_6$ Consider an infinite line charge along z-axis. Show the work done is zero if a point charge $Q$ is moving in a circular path of radius $\rho$ central at the line charge.

**Q.Q ( محاضرات د.أحمد ونفس مثال ص76**

$Q_7$ A point charge of 5 $nC$ is located at the origin. $If\ V = 2\ V$ at $(0, 6, -8)$, find

(a) The potential at $A(-3, 2,6)$

(b) The potential at $B(1, 5, 7)$

(c) The potential difference $V_{AB}$

***Answer*:** (a) $3.929\ V$, (b) $2.696\ V$, (c) $-1.233\ V$. Sadeqo p(138)

$Q_8$ If three point charges, $3\mu C, -4\mu C\ and\ 5\mu C$ are located at $(0, 0, 0), (2, -1, 3) and\ (0, 4, -2)$ respectively. Find the potential at (-1, 5, 2) assuming $V(\infty) = 0$ ***Answer*** $10.23\ K$v

# CONDUCTORS AND DIELECTRICS

## 4.1 Current and Current Density

Electric charges in motion constitute a ***current***. The unit of current is the ampere ($A$), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by $I$, and therefore

$$I = \frac{dQ}{dt}$$

We find the concept of ***current density***, measured in amperes per square meter ($A/m^2$), more useful. Current density is a vector represented by $\mathbf{J}$.

***Total current*** is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot dS$$

Current density may be related to the ***velocity*** of ***volume charge density*** at a point.

$$\mathbf{J} = \rho_v \, \mathbf{U}$$

Where is $\mathbf{U}$ velocity and is $\rho_v$ volume charge density

This last result shows clearly that charge in motion constitutes a current. We call this type of current ***convection current***

***Example:*** Given the current density $\mathbf{J}=10\rho^2 z\, \mathbf{a}_\rho - 4\rho\cos^2\emptyset\ \mathbf{a}_\emptyset$ mA/m$^2$: determine the total current flowing outward through the **circular** band $\rho = 3,\ 0 < \emptyset < 2\pi,\quad 2 < z < 2.8$.?

***Solution:***

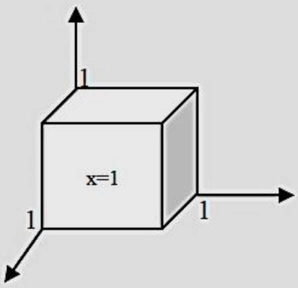
$$I = \int_S \mathbf{J}.dS = \int\int (10\rho^2 z\, \mathbf{a}_\rho - 4\rho\cos^2\emptyset\ \mathbf{a}_\emptyset).\rho\, d\emptyset\, dz\, \mathbf{a}_\rho$$

$$I = \int_2^{2.8}\int_0^{2\pi} 10\rho^3 z\ d\emptyset\, dz = 10(3)^2 \times 2\pi \times \left[\frac{z^2}{2}\right]_2^{2.8} = 3.26\, mA$$

-

***Example:*** *Find the total current outward directed from a 1m cube with one corner at the origin and edge parallel to the coordinate axes if* $\mathbf{J}=2x^2\mathrm{a}_x + 2xy^3\mathrm{a}_y + 2xy\,\mathrm{a}_z$ A/m$^2$?

***Solution:***




$$I = \int_S \mathbf{J}.dS$$

$$dS = dx\,dy\mathrm{a}_z + dy\,dz\mathrm{a}_x + dz\,dx\,\mathrm{a}_y$$

$$I = \iint (2x^2\mathrm{a}_x + 2xy^3\mathrm{a}_y + 2xy\,\mathrm{a}_z).(dx\,dy\mathrm{a}_z + dy\,dz\mathrm{a}_x + dz\,dx\,\mathrm{a}_y)$$

$$I = \iint 2x^2 dy\,dz \quad + \iint 2xy^3 dz\,dx \quad + \iint 2xy\,dx\,dy$$

$(at\ x = 1) - (at\ x = 0)$ $\quad (at\ y = 1) - (at\ y = 0)$ $\quad (at\ z = 1) - (at\ z = 0)$

$$I = \int_0^1\int_0^1 2\,dy\,dz - 0 + \int_0^1\int_0^1 2x\,dz\,dx - 0 + \int_0^1\int_0^1 2xy\,dx\,dy - \int_0^1\int_0^1 2xy\,dx\,dy$$

$$I = 2[y]_0^1\,[z]_0^1 + [x^2]_0^1\,[z]_0^1 - 0 = 2 + 1 = 3\,A$$

## 4.2 Continuity of Current

The continuity equation follows when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \int_S \mathbf{J} \cdot dS$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by $Q_i$ , then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \int_S \mathbf{J} \cdot dS = \frac{-dQ_i}{dt}$$

$$\int_S \mathbf{J} \cdot dS = \int_{vol} (\nabla \cdot \mathbf{J})\, dv \qquad *****$$

We next represent the enclosed charge $Q_i$ by the volume integral of the charge density

$$\int_{vol} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{vol} \rho_v\, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral

$$\int_{vol} (\nabla \cdot \mathbf{J})\, dv = \int_{vol} \frac{-\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$\nabla \cdot \mathbf{J} = \frac{-\partial \rho_v}{\partial t}\ dv$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

## 4.3 Metallic Conductors

Let us first consider the conductor. Here the valence electrons, or *conduction,* or *free,* electrons, move under the influence of an electric field. With a field **E**, an electron having a charge $Q = -e$ will experience a force

$$\vec{\mathbf{F}} = -e\,\vec{\mathbf{E}}$$

In free space, the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material, the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity $\mathbf{U}_d$ is termed the ***drift velocity***, and it is linearly related to the electric field intensity by the ***mobility*** of the electron in the given material.

$$\boldsymbol{U}_d = \mathbf{E}\,\boldsymbol{\mu}_e$$

Where $\boldsymbol{\mu}_e$ is the mobility of an electron, mobility is measured in the units of square meters per volt-second; typical values are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

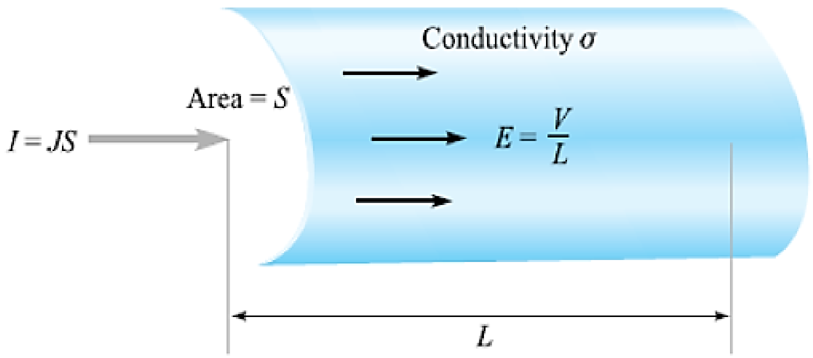
$$\boldsymbol{U}_d = -\rho_e\,\mathbf{E}\,\boldsymbol{\mu}_e$$

where $\rho_e$ is the free-electron charge density

The relationship between **J** and **E** for a metallic conductor, however, is also specified by the conductivity $\boldsymbol{\sigma}$ (sigma),

$$\boldsymbol{\sigma} = -\rho_e \ \boldsymbol{\mu}_e$$

If a conductor of uniform cross-sectional area S and length L, as shown in Figure below, has a voltage difference V between its ends, assume that **J** and **E** are *uniform*




$$I = \int_S \boldsymbol{J}.d\boldsymbol{S} = JS \quad \Rightarrow \quad J = \frac{1}{S} = \boldsymbol{\sigma E}$$

$$V_{AB} = -\int_B^A E.dL = EL \quad \Rightarrow \quad E = \frac{V}{L}$$

$$\therefore \frac{I}{S} = \sigma \ \frac{V}{L}$$

$$\frac{V}{I} = \sigma \ \frac{L}{\sigma S}$$

The ratio of the ***potential difference*** between the ***two ends of the cylinder*** to the current entering the more positive end,

$$R = \sigma \frac{L}{\sigma S}$$

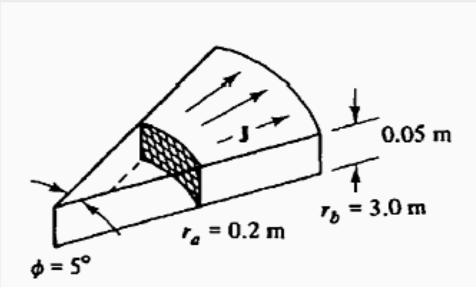
• When the fields are nonuniform,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a E \cdot dL}{\int_S \sigma E \cdot dS}$$

***Example:*** Find the resistance between the inner and outer curved surfaces of the block shown in Fig. below, where the material is silver for which $\sigma = 6.17 \times 10^7$ S/m. if $\mathbf{J} = k/\rho\, \mathbf{a}_\rho$ ?

***Solution:***

$$\mathbf{J} = \sigma\, \mathbf{E}$$




$$E = \frac{\mathbf{J}}{\sigma} = \frac{k}{\sigma\,\rho}\mathbf{a}_\rho$$

$$R = \frac{\int_{0.2}^{3} \frac{k}{\sigma\,\rho}\, d\rho}{\int_0^{0.05}\int_0^{5^o} \frac{k}{\rho}\, \rho d\emptyset dz}$$

$$R = \frac{\ln\frac{3}{0.2}}{\sigma \times 0.0873 \times 0.005}$$

$$R = 1.01 \times 10^{-5}\Omega$$

## 4.4 <u>Conductor Properties and Boundary Conditions</u>

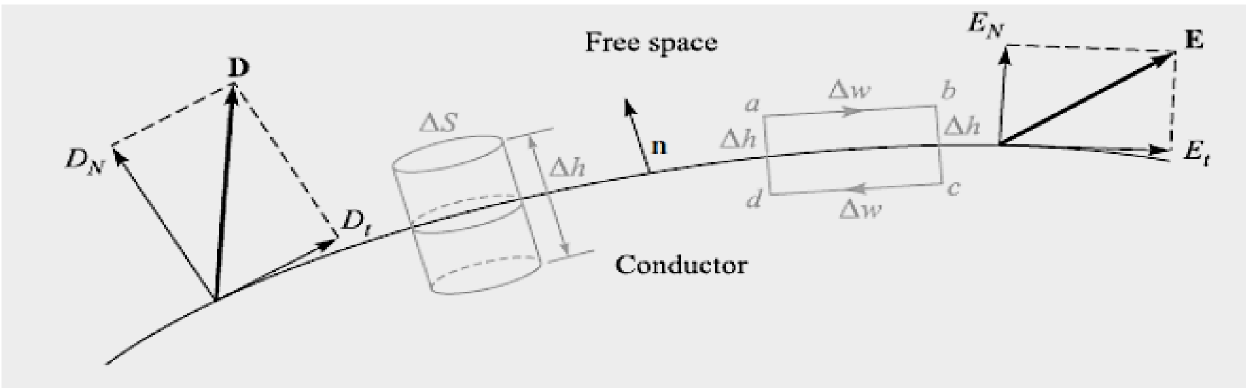
For electrostatics, no charge and no electric field may exist at any point within a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields external to the conductor.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

a. The static electric field intensity inside a conductor is zero.
b. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
c. The conductor surface is an equipotential surface.

## 4.4.1 Conductor-Dielectric Boundary Conditions

Under static conditions all net charge will be on the outer surfaces of a conductor and both $\boldsymbol{E}$ and $\boldsymbol{D}$ are therefore zero within the conductor. Because the electric field is a conservative field, the line integral of $\boldsymbol{E}.d\boldsymbol{L}$ is zero for any closed path. A rectangular path with corners 1, 2, 3, 4 is shown in Figure below.




$$\oint E \cdot dL = 0$$

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

If the path lengths b to c and d to a are now permitted to approach zero, keeping the interface between them, then the second and fourth integrals are zero.

The path from $c$ to $d$ is within the conductor where E must be zero. This leaves

$$\int_a^b E \cdot dL = \int_a^b E_t \, dl$$

where $E_t$ is the tangential component of E at the surface of the dielectric. Since the interval a to b can be chosen arbitrarily, at each point of the surface.

$$E_t = D_t = 0$$

To discover the conditions on the normal components, a small, closed, right circular cylinder is placed across the interface; Gauss' law applied to this surface gives

$$\oint D \cdot dS = Q_{enc}$$

$$\int_{top} D \cdot dS + \int_{bottom} D \cdot dS + \int_{side} D \cdot dS = \int_A \rho_S \cdot dS$$

The third integral is zero since, as just determined, $D_t = 0$ on either side of the interface. The second integral is also zero, since the bottom of the cylinder is within the conductor, where D and E are zero. Then,

$$\int_{top} D \cdot dS = \int_{top} D_N \, dS = \int_A \rho_S \, dS$$

$$D_N = \rho_S$$

$$D_N = \varepsilon_0 E_N = \rho_S$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density.

***Example:*** A solid conductor has a surface described by x + y=3m and extends toward the origin. At the surface the electric field intensity is 0.35 V/m. Express E and D at the surface and find $\rho_s$?

***Solution:***

The unit vector normal to the surface is :

$$a_N = \frac{a_x + a_y}{\sqrt{2}}$$

$$E_N = 0.35 \frac{a_x + a_y}{\sqrt{2}} = 0.247(a_x + a_y)$$

$$D_N = \varepsilon_o E_N = 2.19 \times 10^{-12}(a_x + a_y)$$

$$\rho_s = |D_N| = 3.09 \times 10^{-12}$$

## 4.5 The Nature of Dielectric Materials

A dielectric in an electric field can be viewed as a ***free-space arrangement of microscopic electric dipoles***, each of which is composed of a positive and a negative charge whose centers do not quite coincide. These are ***not free charges,*** and they cannot ***contribute to the conduction process.*** Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound* charges.

The characteristic that all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their **ability to store electric energy**. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

The dipole may be described by its dipole moment **P**

$$\mathbf{P} = Q\boldsymbol{d}$$

where $Q$ is the positive one of the two bound charges composing the dipole, and $\boldsymbol{d}$ is the vector from the negative to the positive charge. We note again that the units of **P** are coulomb-meters.
There is thus an added term to **D** that appears when polarizable material is present

$$D = \varepsilon_0 \mathrm{E} + \mathbf{P}$$

The linear relationship between **P** and **E** is

$$\mathbf{P} = \chi_e \varepsilon_0 \mathrm{E}$$

where $\chi_e$ is a dimensionless quantity called the *electric susceptibility* of the material.

$$D = \varepsilon_0 \mathrm{E} + \mathbf{P} = \varepsilon_0 \mathrm{E} + \chi_e \varepsilon_0 \mathrm{E}$$

$$D = (\chi_e + 1)\varepsilon_0 \mathrm{E}$$

The expression within the parentheses is now defined as

$$\varepsilon_r = \chi_e + 1$$

This is another dimensionless quantity, and it is known as the ***relative permittivity***, or ***dielectric constant*** of the material. Thus

$$D = \varepsilon_r \, \varepsilon_0 \mathrm{E} = \varepsilon \mathrm{E}$$

$$\varepsilon = \varepsilon_r \, \varepsilon_0$$

where $\varepsilon$ is the permittivity

***Example*** *Two point charges in a dielectric medium where* $\varepsilon_r = 5.2$ *interact with a force of* $8.6 \times 10^{-3}$ *N. What force could be expected if the charges were in free space??*

***Solution:***

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \qquad in\ free\ space$$

$$F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon\, R^2} \qquad in\ dielectric$$

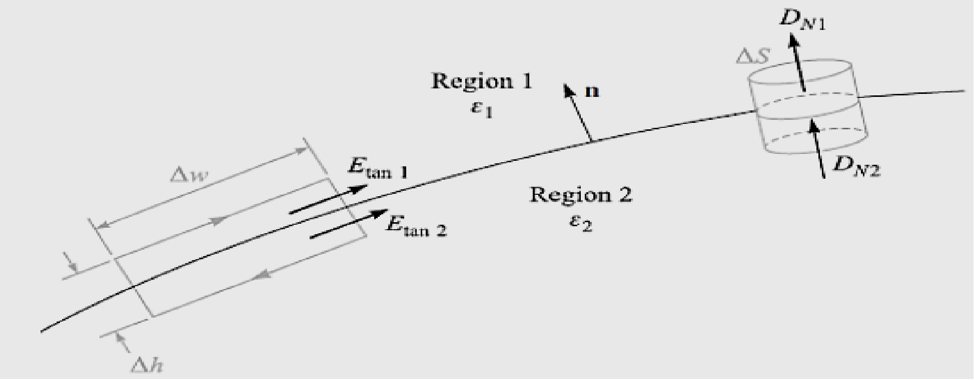
$$\frac{F_1}{F_2} = \frac{\varepsilon}{\varepsilon_o}$$

$$F_1 = \frac{\varepsilon_o \varepsilon_r}{\varepsilon_o} F_2 = \varepsilon_r F_2 = 5.2 \times 8.6 \times 10^{-3} = 4.47 \times 10^{-2}\ N$$

## 4.6 Boundary Conditions for Perfect Dielectric Materials

Let us first consider the interface between two dielectrics having permittivities and and occupying regions 1 and 2, as shown in Figure below. We first examine the tangential components by using

$$\oint E \cdot dL = 0$$




Around the small closed path on the left, obtaining

$$E_{tan_1}\Delta w - E_{tan_2}\Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length $\Delta h$ becomes negligible as $\Delta h$ decreases and the closed path crowds the surface. Immediately, then

$$E_{tan_1} = E_{tan_2}$$

$$\frac{D_{tan_1}}{\varepsilon_1} = \frac{D_{tan_2}}{\varepsilon_2}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in the Figure. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N_1}\Delta S - D_{N_2}\Delta S = \Delta Q = \rho_S\,\Delta S$$

for no free charge is available in the perfect dielectrics, we may assume $\rho_S$ is zero on the interface and

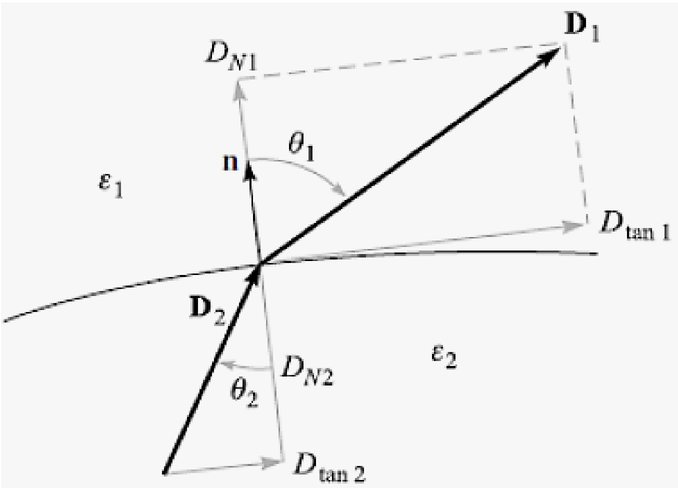
$$D_{N_1} = D_{N_2}$$

Let $\mathbf{D}_1$ (and $\mathbf{E}_1$) make an angle with $\theta_1$a normal to the surface (Figure below) . Because the normal components

of **D** are continuous,

$$D_{N_1} = D_{N_2}$$

$$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2$$



The ratio of the tangential components is given by

$$\frac{D_{tan_1}}{D_{tan_2}} = \frac{\varepsilon_1}{\varepsilon_2} \qquad \Leftrightarrow \qquad \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

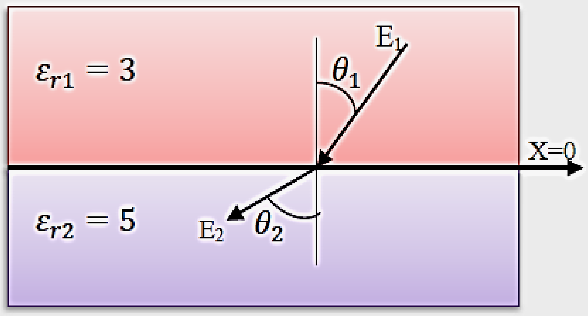
$$D_1\, \varepsilon_2 \sin\theta_1 = D_2\, \varepsilon_1 \sin\theta_2$$

and the division of these equations gives

$$\frac{tan\ \theta_1}{tan\ \theta_1} = \frac{\varepsilon_1}{\varepsilon_2}$$

***Example*** *The surface x = 0 separates two perfect dielectrics. For x > 0, let* $\varepsilon_{r1} = 3$, *while* $\varepsilon_{r2} = 5$ *where x < 0. If* $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ *V/m, find (a)* $E_{N1}$*; (b)* $\mathbf{E}_{T1}$ *; (c)* $\mathbf{E}_1$*; (d) the angle* $\theta_1$ *between* $\mathbf{E}_1$ *and a normal to the surface; (e)* $D_{N2}$*; (f)* $D_{T2}$*; (g)* $\mathbf{D}_2$*; (h)* $\mathbf{P}_2$*; (i) the angle* $\theta_2$ *between* $\mathbf{E}_2$ *and a normal to the surface.?*

***Solution:***




*(a)* $E_{N1} = 80\ ax$

*(b)* $E_{T1} = -60ay - 30az$

*(c)* $|E_1| = \sqrt{80^2 + 60^2 + 30^2} = 104.4\ V/m$

*(d)* $E_{N1} = E_1 \cos\theta_1$

$$\cos\theta_1 = \frac{E_{N1}}{E_1} = \frac{80}{104.4}$$

$$\theta_1 = 40^0$$

*(e)* $D_{N2} = D_{N1}$

$$D_{N1} = \varepsilon_1 E_{N1} = \varepsilon_0 \varepsilon_{r1} E_{N1} = 3 \times 8.85 \times 10^{-12} \times 80 \text{ a}x = 2.12 \text{ a}x \ \ nC/m^2$$

*(f)* $E_{T1} = E_{T2}$

$$\frac{D_{T1}}{\varepsilon_1} = \frac{D_{T2}}{\varepsilon_2}$$

$$D_{T2} = \frac{\varepsilon_2}{\varepsilon_1} D_{T1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} \ \varepsilon_1 E_{T1} = \varepsilon_{r2} \varepsilon_0 E_{T1} = \varepsilon_{r2} \varepsilon_0 (-60\text{ay} - 30\text{az})$$

$$D_2 = D_{N2} + D_{T2}$$

$$D_2 = 2.12 \times 10^{-9} \text{ a}_x - 60 \ \varepsilon_{r2} \varepsilon_0 \text{a}_y - 30 \ \varepsilon_{r2} \varepsilon_0 \text{ a}_z$$

*(h)* $P_2 = (\varepsilon_{r2} - 1) \ \varepsilon_0 E_2$

$$= 4 \ \varepsilon_0 \frac{D_2}{\varepsilon_2}$$

$$= 4 \ \varepsilon_0 \frac{D_2}{\varepsilon_2}$$

$$= 4 \ \varepsilon_0 \frac{D_2}{\varepsilon_0 \ \varepsilon_{r2}}$$

$$= 4 \frac{D_2}{\varepsilon_{r2}}$$

$$= \frac{4}{5}(2.12 \times 10^{-9} \text{ a}_x - 60 \ \varepsilon_{r2} \varepsilon_0 \text{a}_y - 30 \ \varepsilon_{r2} \varepsilon_0 \text{ a}_z)$$

*(i)*

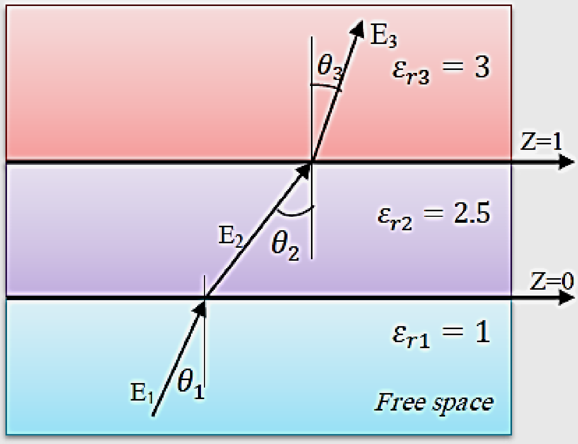
$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\frac{\tan 40}{\tan \theta_2} = \frac{3}{5}$$

$$\theta_2 = 54.5^0$$

***Example*** *Region 1, z<0 m, is free space where* $D = 5\mathbf{a}_y + 7\mathbf{a}_z$ *C/m*$^2$. *Region 2, 0< z <1 m, has* $\varepsilon_r = 2.5$ . *And region 3, z > 1 m, has* $\varepsilon_r = 3$ *Find* $E_2$, $\mathbf{P}_2$ , *and* $\theta_3$?

***Solution:***




$$D_1 = (5\mathrm{a}_y + 7\,\mathrm{a}_z)$$

$$D_{N1} = D_1 \cos\theta_1$$

$$\cos\theta_1 = \frac{7}{\sqrt{5^2 + 7^2}}$$

$$\theta_1 = 35.53^0$$

$$D_{N2} = D_{N1} = 7\,\mathrm{a}_z$$

$$E_{T1} = E_{T2}$$

$$\frac{D_{T1}}{\varepsilon_1} = \frac{D_{T2}}{\varepsilon_2}$$

$$D_{T2} = \frac{\varepsilon_2}{\varepsilon_1} D_{T1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} D_{T1} = \frac{2.5}{1} 5\mathrm{a}_y = 12.5\,\mathrm{a}_y$$

$$D_2 = D_{N2} + D_{T2}$$

$$D_2 = 12.5\,\mathrm{a}_y + 7\,\mathrm{a}_z$$

$$E_2 = \frac{D_2}{\varepsilon_2} = \frac{12.5\,\mathrm{a}_y + 7\,\mathrm{a}_z}{2.5\,\varepsilon_0} = \frac{1}{\varepsilon_0}\left(5\,\mathrm{a}_y + \frac{7}{2.5}\,\mathrm{a}_z\right)$$

$$D_{N2} = D_2 \cos\theta_2$$

$$\theta_2 = \cos^{-1}\frac{7}{\sqrt{12.5^2 + 7^2}} = 60.7^0$$

$$\frac{\tan\theta_2}{\tan\theta_3} = \frac{\varepsilon_2}{\varepsilon_3}$$

$$\tan\theta_3 = \frac{\varepsilon_{r3}}{\varepsilon_{r2}}\tan\theta_2 = \frac{3}{2.5}\tan 60.7$$

$$\theta_3 = 64.93^0$$

## 4.7 Poisson's and Laplace's Equations

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium)

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \rho_v$$

and

$$\mathbf{E} = -\nabla V$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

Equation above is ***Poisson's equation***,

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0$$

which is ***Laplace's equation***. The $\nabla^2$ operation is called the *Laplacian of V*.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad \text{( Cartesian)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{( cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \emptyset^2} \qquad \text{( spherical)}$$

***Example***

Determine whether or not the following potential fields satisfy the Laplace's equation

a) $V = x^2 - y^2 + z^2$ b) $V = \rho \cos \emptyset + z$

***Solution***

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial y^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial z^2}[x^2 - y^2 + z^2]$$

$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] + \frac{\partial}{\partial z}[2z] = 2$$

So $\nabla^2 V \neq 0$

Hence field $V$ **does not satisfy Laplace's equation.**

b)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial V}{\partial \rho} = \frac{\partial}{\partial \rho}[\rho \cos \emptyset + z] = \cos \emptyset$$

$$\frac{\partial V}{\partial \emptyset} = \frac{\partial}{\partial \emptyset}[\rho \cos \emptyset + z] = -\rho \sin \emptyset$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[\rho \cos \emptyset + z] = 1$$

$$\therefore \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho \cos \emptyset] = \frac{1}{\rho}\cos \emptyset$$

$$\frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right) = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \emptyset}(-\rho \sin \emptyset)\right] = -\frac{\rho \cos \emptyset}{\rho^2} = -\frac{\cos \emptyset}{\rho}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}[1] = 0$$

$$\nabla^2 V = \frac{1}{\rho}\cos\emptyset - \frac{\cos\emptyset}{\rho} + 0 = 0$$

So this field **satisfies Laplace's equation.**

## *Home work*

$\boldsymbol{Q_1}$: Find the Polarization in dielectric material with $\varepsilon_r = 2.8$ if

$\vec{D} = 3 \times 10^{-7}\ C/m^2$ ***Answer*** $1.92 \times 10^{-7}\ C/m^2$

$\boldsymbol{Q_2}$: Find the magnitude of $\vec{D}$ and $\vec{P}$ for dielectric material in which

$|\vec{E}| = 0.15\ mV/m$ and $\chi_e = 4.25$.

***Answer*** $|\vec{D}| = 6.9725 \times 10^{-15}\ C/m^2$ $|\vec{P}| = 5.644 \times 10^{-15}\ C/m^2$

$\boldsymbol{Q_3}$: The region with $z < 0$ is characterized by $\varepsilon_{r2} = 2$ and $z > 0$ by $\varepsilon_{r1} = 5$.

If $\vec{D}_1 = 2\,\hat{a}_x + 5\,\hat{a}_y - 3\,\hat{a}_Z\ (n\mathrm{C/m^2})$, find :

a) $\vec{D}_2$ b) $\vec{D}_{2n}$ c) $\vec{D}_{2t}$

d) the angle that $\vec{D}_2$ makes with z axis e) $\frac{|\vec{D}_2|}{|\vec{D}_1|}$

***Answer*** a) $0.8\,\hat{a}_x + 2\,\hat{a}_y - 3\,\hat{a}_Z\ (n\mathrm{C/m^2})$ b) $-3\,\hat{a}_Z$ c) $0.8\,\hat{a}_x + 2\,\hat{a}_y$

d) $35.678°$ e) $0.599$

# The Steady Magnetic Field

## 5.1 BIOT-SAVART Law

The source of the steady magnetic field may be a ***permanent magnet***, an ***electric field changing linearly with time***, or a ***direct current***. We will largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present study will concern the magnetic field produced by a differential dc element in free space

**Biot-Savart's law** states that ***"the magnetic field intensity* dH *produced at a point P by the differential current element I dl is proportional to the product I dl and the sine of the angle between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element".***

The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point *P* as shown in Figure 5.1.

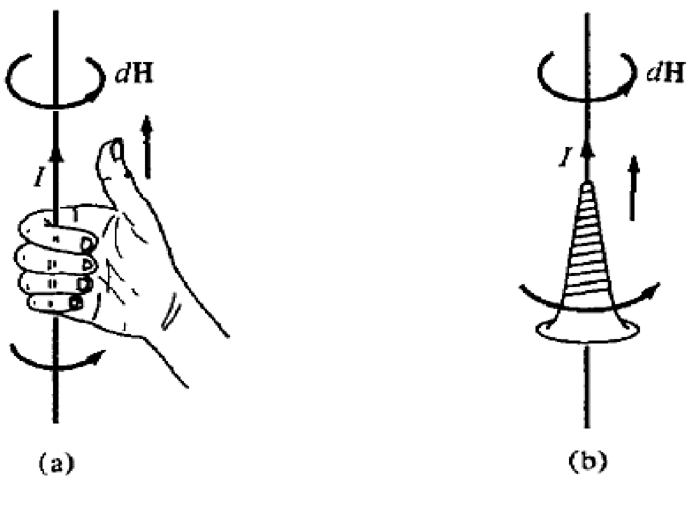



**Figure 5.1** the direction of dH using (a) the right-hand rule, or (b) the right-handed screw rule.

It is customary to represent the direction of the magnetic field intensity H (or current $I$) by a small circle with a dot or cross sign depending on whether H (or $I$) is out of, or into, the page as illustrated in Figure 5.2.


**Figure 5.2** Conventional representation of H (or *I)* (a) out of the page and (b) into the page.

We can have different current distributions: line current, surface current, and volume current. If we define K as the surface current density (in $A/m$) and J as the volume current density (in $A/m^2$),

$$H = \oint \frac{I\, dL \times \mathrm{a}_R}{4\pi R^2} \qquad \text{Line current}$$

$$H = \oint \frac{K\, dS \times \mathrm{a}_R}{4\pi R^2} \qquad \text{Surface current}$$

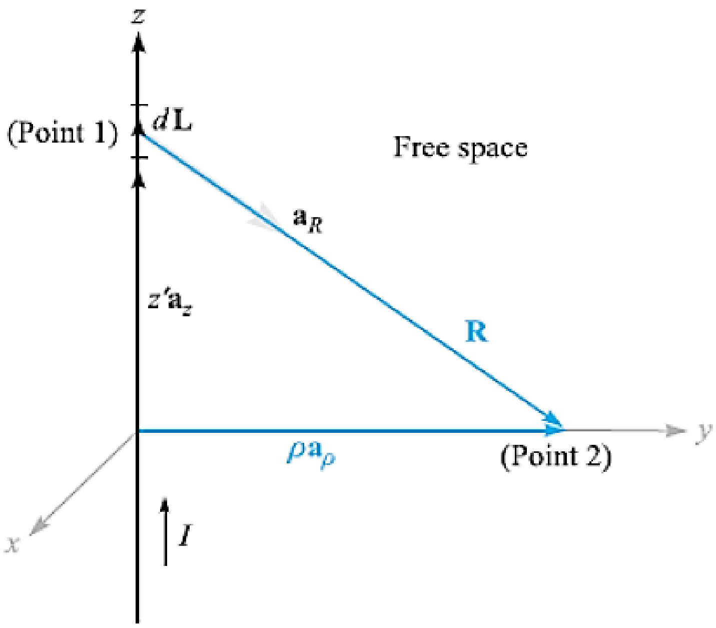
$$H = \oint \frac{\mathrm{J}\, dv \times \mathrm{a}_R}{4\pi R^2} \qquad \text{Volume current}$$

Consider an ***infinitely long straight filament*** carrying a direct current I is located along z-axis

$$H = \oint \frac{I\, dL \times \mathrm{a}_R}{4\pi R^2}$$

$$dL = dz\mathbf{a}_z$$

$$R = \rho \mathbf{a}_\rho - z\,\mathbf{a}_z$$




$$|R| = \sqrt{\rho^2 + z^2}$$

$$H = \int_{-\infty}^{\infty} \frac{I\, dz \mathbf{a}_z}{4\pi\left(\sqrt{\rho^2 + z^2}\right)^2} \times \frac{\rho \mathbf{a}_\rho - z\,\mathbf{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$H = \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\, dz}{(\rho^2 + z^2)^{\frac{3}{2}}}\,\mathbf{a}_\emptyset \quad , \quad (\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\emptyset) \quad , \quad (\mathbf{a}_z \times \mathbf{a}_z = 0)$$

$Let\ z = \rho\ \tan u$ , $\qquad dz = \rho \sec^2 u\ du$

$$u = \tan^{-1}\frac{z}{\rho} \qquad u_1 = \tan^{-1}\frac{-\infty}{\rho} = \frac{-\pi}{2}, \qquad u_2 = \tan^{-1}\frac{\infty}{\rho} = \frac{\pi}{2}$$

$$H = \frac{I}{4\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\rho\,\rho \sec^2 u\ du}{(\rho^2 + \rho^2 \tan^2 u)^{\frac{3}{2}}}\,\mathbf{a}_\emptyset$$

$$H = \frac{I}{4\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\rho\,\rho \sec^2 u\ du}{(\rho^2 + \rho^2 \tan^2 u)^{\frac{3}{2}}}\,\mathbf{a}_\emptyset$$

$$H = \frac{I}{4\pi\rho}[\sin u]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \frac{I}{4\pi\rho}\left[\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right]$$

$$H = \frac{I}{2\pi\rho}\mathbf{a}_{\emptyset} \qquad \textit{in cylindrical}$$

$$H = \frac{I}{2\pi\,\rho}\mathbf{a}_{\emptyset} \quad = \frac{I}{2\pi}\left(\frac{-y\mathrm{a}_x + x\mathrm{a}_y}{x^2+y^2}\right) \qquad \textit{in cartesian along z axis}$$

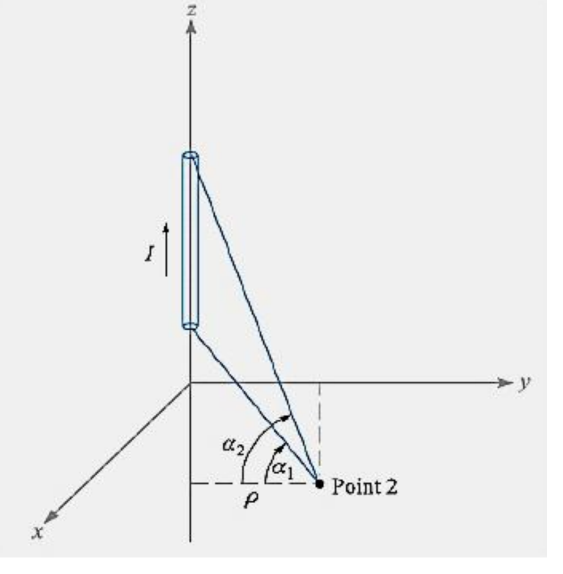
The finite-length current element is shown in Figure below. The magnetic field intensity H is most easily expressed in terms of the angles $\alpha_1$ and $\alpha_2$, as identified in the figure. The result is

$$H = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)a_{\emptyset}$$

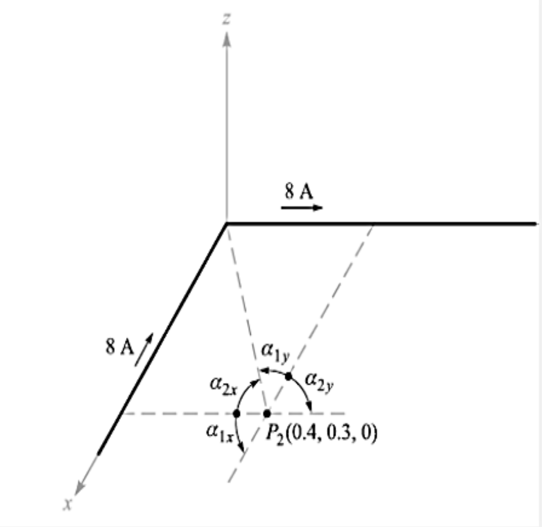
To find unit vector $a_{\emptyset}$ in eqs. above, a simple approach is use to determine it from

$$a_{\emptyset} = a_l + a_\rho$$

where $a_l$ is a unit vector along the line current and $a_\rho$ is a unit vector along the perpendicular line from the line current to the field point.



***Example:*** Determine **H** at *P* (0.4, 0.3, 0) in the field of an 8 A filamentary current is directed inward from infinity to the origin on the positive *x* axis, and then outward to infinity along the *y* axis. As shown in Figure.




***Solution:***

$$H = H_1 + H_2$$

$$H_1 = \frac{I_1}{4\pi\rho_1}(\sin\alpha_2 - \sin\alpha_1)$$

$$\rho_1 = 0.3 \quad , \quad \alpha_1 = -90 \quad , \quad \alpha_2 = \tan^{-1}\frac{0.4}{0.3} = 53.1$$

$$H_1 = \frac{8}{4\pi 0.3}(\sin 53.1 + \sin 90)\mathbf{a}_\emptyset = \frac{12}{\pi}\mathbf{a}_\emptyset = \frac{-12}{\pi}\mathbf{a}_z$$

$$H_2 = \frac{I_2}{4\pi\rho_2}(\sin\alpha_2 - \sin\alpha_1)$$

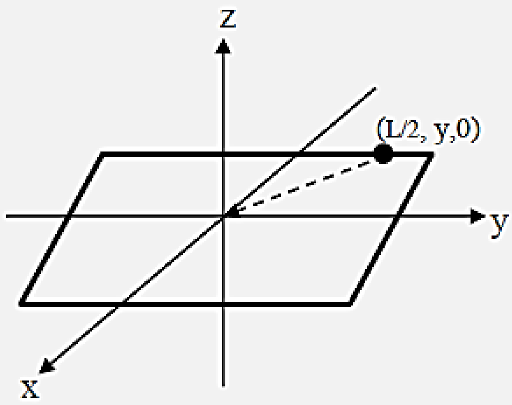
$$\rho_2 = 0.4 \quad , \quad \alpha_2 = 90 \quad , \quad \alpha_1 = -\tan^{-1}\frac{0.3}{0.4} = -36.9$$

$$H_2 = \frac{8}{4\pi 0.4}(\sin 90 + \sin 36.9)\mathbf{a}_\emptyset = \frac{-8}{\pi}\mathbf{a}_\emptyset = \frac{8}{\pi}\mathbf{a}_z$$

$$H = H_1 + H_2 = \frac{-12}{\pi}\mathbf{a}_z + \frac{8}{\pi}\mathbf{a}_z = \frac{-20}{\pi}\mathbf{a}_z$$

***Example:*** Find H at the center of a square current loop of side L is located on xy-plane?

***Solution:***




$$H = \oint \frac{I\,dL \times \mathbf{a}_R}{4\pi R^2}$$

$$dL = dy\,\mathbf{a}_y$$

$$R = \frac{-L}{2}\mathbf{a}_x - y\,\mathbf{a}_y$$

$$|R| = \sqrt{\left(\frac{L}{2}\right)^2 + y^2}$$

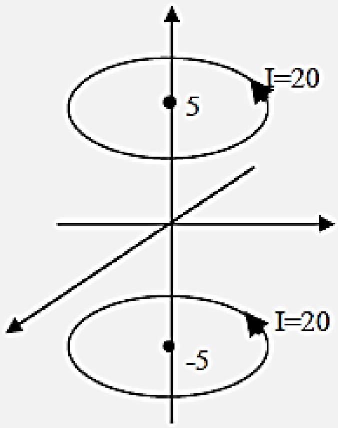
$$H_1 = \int \frac{I\,dy\,\mathbf{a}_y}{4\pi\left(\sqrt{\left(\frac{L}{2}\right)^2 + y^2}\right)^2} \times \frac{\frac{-L}{2}\mathbf{a}_x - y\,\mathbf{a}_y}{\sqrt{\left(\frac{L}{2}\right)^2 + y^2}}$$

$$H_1 = \frac{I}{4\pi}\int_0^{\frac{L}{2}} \frac{\frac{L}{2}\;dy}{\left(\left(\frac{L}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\;\mathbf{a}_z \qquad , \quad (\mathbf{a}_y \times \mathbf{a}_x = -\mathbf{a}_z) \;\; , \quad (\mathbf{a}_y \times \mathbf{a}_y = 0)$$

$$H_1 = \frac{\sqrt{2}\,I}{4\pi L}\mathbf{a}_z$$

$$H = 8\,H_1 = 8 * \frac{\sqrt{2}\,I}{4\pi L}\mathbf{a}_z = \frac{2\sqrt{2}\,I}{\pi L}\mathbf{a}_z$$

***Example:*** Two identical circular current loops of radius $\rho = 3$ and I = 20A are in parallel planes, separated on their common axis by 10 m. Find H at a point midway between the two loops?




***Solution:***

$H = H_1 + H_2$

$H_1 = \oint \frac{I\, dL_1 \times \mathbf{a}_{R1}}{4\pi {R_1}^2}$

$dL_1 = \rho d\emptyset \mathbf{a}_\emptyset = 3d\emptyset \mathbf{a}_\emptyset$

$R_1 = -3\mathbf{a}_\rho - 5\,\mathbf{a}_z$

$|R| = \sqrt{3^2 + 5^2} = \sqrt{34}$

$$H_1 = \int_0^{2\pi} \frac{I * 3d\emptyset \mathbf{a}_\emptyset}{4\pi * 34} \times \frac{-3\mathbf{a}_\rho - 5\,\mathbf{a}_z}{\sqrt{34}} = \frac{3I}{4\pi * 34^{\frac{3}{2}}}\left[\int_0^{2\pi} 3d\emptyset \mathbf{a}_z + \int_0^{2\pi} -5d\emptyset \mathbf{a}_\rho\right] = 0.453\mathbf{a}_z$$

$H_2 = H_1$ , $H = 0.908\mathbf{a}_z$

## 5.2 AMPERE'S Circuital Law

Ampere's circuital law states that "***the line integral of H about any closed path is exactly equal to the direct current enclosed by that path***"

$$\oint \mathrm{H}.\, dL = I_{enc}$$

We choose a path, to any section of which **H** is either ***perpendicular*** or ***tangential***, and along which $H$ is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampere's circuital law with the product of the scalar magnitudes, except along that portion of the path where **H** ***is normal to the path and the dot product is zero***; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which **H** is parallel.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current $I$. The filament lies on the $z$ axis in free space, and the current flows in the direction given by $\boldsymbol{a_z}$ .

The path must be a circle of radius $\rho$ , and Ampere's circuital law becomes

$$\oint \mathrm{H}.\, dL = I_{enc}$$

$$\oint \mathrm{H}.\, dL = \int_0^{2\pi} \mathrm{H}_\emptyset\, \rho d\emptyset \ = \mathrm{H}_\emptyset\, \rho \int_0^{2\pi} d\emptyset = 2\pi\, \mathrm{H}_\emptyset\, \rho$$

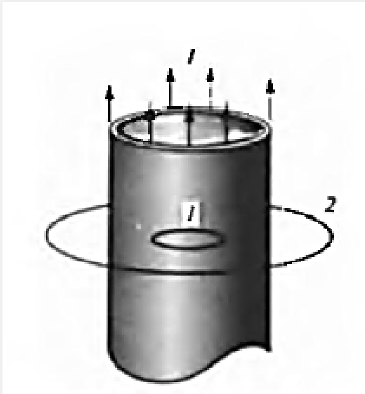
$$2\pi\, \mathrm{H}_\emptyset\, \rho = I$$

$$\mathrm{H}_\emptyset \ = \frac{I}{2\pi\, \rho}$$

$$\boxed{\mathrm{H} = \frac{I}{2\pi\, \rho}\ \mathrm{a}_\emptyset}$$

***Example:*** A thin cylindrical conductor of radius a, infinite in length, carries a current I. Find H at all points using Ampere's law?

***Solution:***




For path 1 inside cylinder

$$\oint \mathrm{H}.\, dL = I_{enc}$$

$$I_{enc} = 0$$

$$\therefore \mathrm{H} = 0$$

For path 2 outside cylinder

$$I_{enc} = I$$

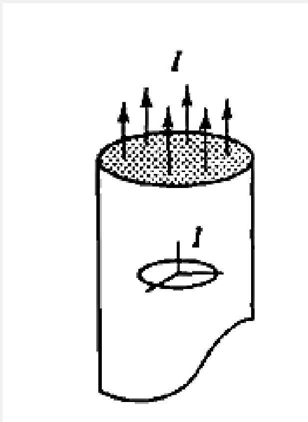
$$\oint \mathrm{H}.\, dL = \int_0^{2\pi} \mathrm{H}_\emptyset\, \rho d\emptyset \; = \mathrm{H}_\emptyset\, \rho \int_0^{2\pi} d\emptyset = 2\pi\rho\, \mathrm{H}_\emptyset$$

$$2\pi\rho\, \mathrm{H}_\emptyset \; = I$$

$$\mathrm{H}_\emptyset \; = \frac{I}{2\pi\, \rho}$$

***Example:*** Determine H for a solid cylindrical conductor of radius a, where the current I is uniformly distributed over the cross section?

***Solution:***




$$for\; \rho < a$$

$$I_{enc} = I\frac{\pi\rho^2}{\pi a^2} = \frac{\rho^2}{a^2}I$$

$$\oint \mathrm{H}.\, dL = \int_0^{2\pi} \mathrm{H}_\emptyset\, \rho d\emptyset \; = \mathrm{H}_\emptyset\, \rho \int_0^{2\pi} d\emptyset = 2\pi\rho\, \mathrm{H}_\emptyset$$

$$2\pi\rho\, \mathrm{H}_\emptyset \; = \frac{\rho^2}{a^2}I$$

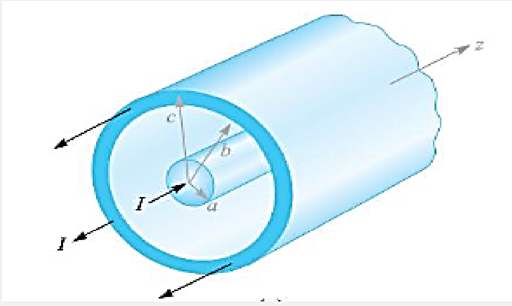
$$\mathrm{H}_\emptyset \; = \frac{I\rho}{2\pi a^2}$$

$$for\; \rho > a \qquad , \qquad I_{enc} = I \qquad , \quad \mathrm{H}_\emptyset \; = \frac{I}{2\pi\, \rho}$$

***Example:*** consider an infinitely long coaxial transmission line carrying a uniformly distributed total current $I$ in the center conductor and $-I$ in the outer conductor, Find H at all points using Ampere's law?

***Solution:***

$for\ \rho < a$




$$I_{enc} = I\frac{\pi\rho^2}{\pi a^2} = \frac{\rho^2}{a^2}I$$

$$\oint \mathrm{H}.\,dL = \int_0^{2\pi} \mathrm{H}_\emptyset\, \rho d\emptyset\ = \mathrm{H}_\emptyset\, \rho \int_0^{2\pi} d\emptyset = 2\pi\rho\, \mathrm{H}_\emptyset$$

$$2\pi\rho\, \mathrm{H}_\emptyset\ = \frac{\rho^2}{a^2}I$$

$$\mathrm{H}_\emptyset\ = \frac{I\rho}{2\pi a^2}$$

$for\ a < \rho < b$

$$I_{enc} = I \qquad , \quad \mathrm{H}_\emptyset\ = \frac{I}{2\pi\, \rho}$$

$for\ b < \rho < c$

$$I_{enc} = I \qquad , \quad \mathrm{H}_\emptyset\ = \frac{I}{2\pi\, \rho}$$

$for\ b < \rho < c$

$$I_{enc} = I_1 + I_2$$

$$I_2 = -I\frac{\pi(\rho^2 - b^2)}{\pi(c^2 - b^2)}$$

$$I_{enc} = I - I\ \frac{\rho^2 - b^2}{c^2 - b^2} = \frac{c^2 - \rho^2}{c^2 - b^2}I$$

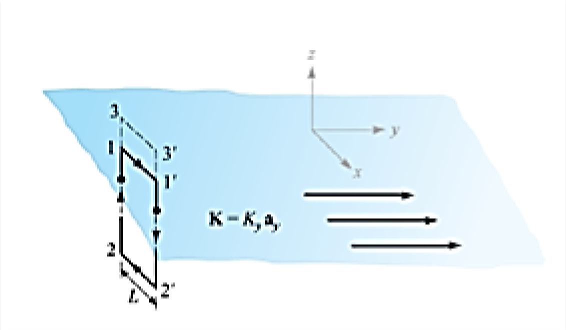
$$2\pi\rho\, \mathrm{H}_\emptyset\ = \frac{c^2 - \rho^2}{c^2 - b^2}I$$

$$\mathrm{H}_\emptyset\ = \frac{I}{2\pi\rho}\ \frac{c^2 - \rho^2}{c^2 - b^2}$$

$for\ c < \rho$

$$I_{enc} = I_1 + I_2\ \ = I - I = 0 \qquad , \therefore\ H = 0$$

***Example:*** consider a ***sheet of current*** flowing in the positive $y$ direction and located in the $z = 0$ plane as shown in Figure below. Find H?




***Solution:***

$$H = H_x a_x + H_y a_y + H_z a_z$$

$$H_y = H_z = 0$$

$$H = H_x a_x$$

$$I_{enc} = K\,L$$

$$\oint \mathrm{H}.\,dL = \int H_x a_x.dx a_x + \int H_x a_x.dz a_z = H_x L + H_x L = 2H_x L$$

$$\oint \mathrm{H}.\,dL = I_{enc}$$

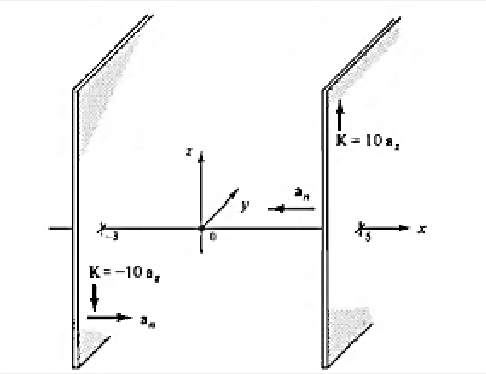
$$2H_x L = K\,L$$

$$H_x = \frac{K}{2}$$

***In general the H for infinite sheet current is given by:***

$$H = \frac{1}{2} K \times a_N$$

***Example:*** A current sheet, **K=10a$_z$** A/m, lies in the **x = 5m** plane and a second sheet, **K = -10a$_z$** A/m, is at **x = -5 m**. Find H at all points?




***Solution:***

$$for\ -5 < x < 5$$

$$H_1 = \frac{1}{2} K_1 \times a_N$$

$$H_1 = \frac{1}{2} 10 a_z \times -a_x = -5a_y$$

$$H_2 = \frac{1}{2} - 10a_z \times a_x = -5a_y$$

$$H = H_1 + H_2 = -10a_y$$

$$for\ x < -5$$

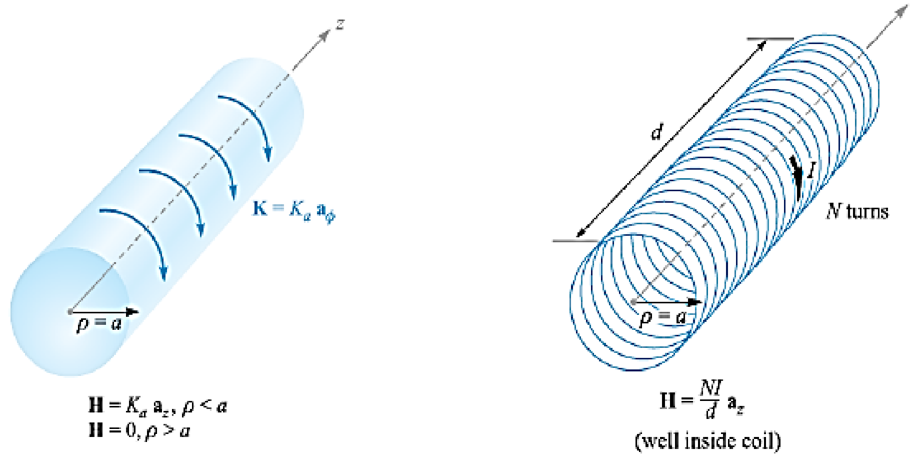
$$H_1 = \frac{1}{2} 10a_z \times -a_x = -5a_y$$

$$H_2 = \frac{1}{2} - 10a_z \times -a_x = 5a_y$$

$$H = H_1 + H_2 = -5a_y + 5a_y = 0$$

(a) An ideal solenoid of infinite length with a circular current sheet K.

(b) An *N*-turn solenoid of finite length *d*.



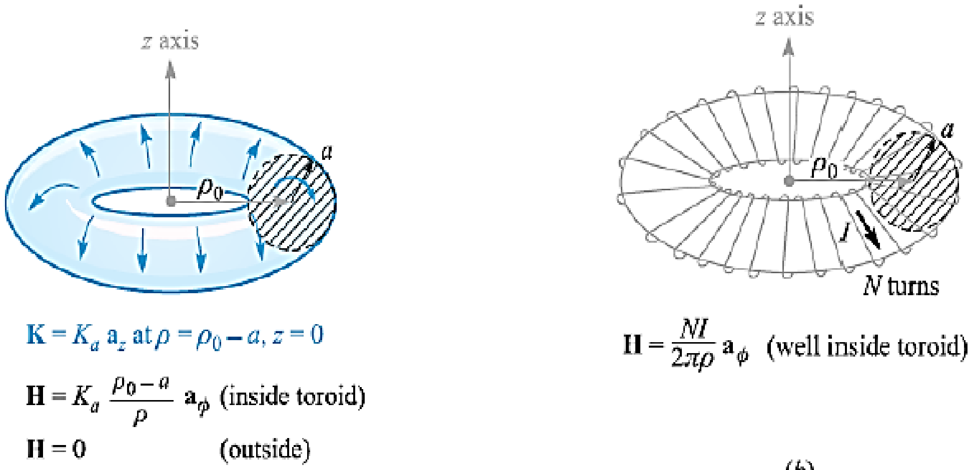

(a) (b)

$$H = K_a \mathbf{a}_z \qquad \rho < a$$

$$H = \frac{N\,I}{d}\mathbf{a}_z \qquad \rho < a$$

$$H = 0 \qquad \rho > a$$

For the toroid shown in Figure below, it can be shown that the magnetic field intensity for the ideal case




(a) (b)