STUDENTS' PERFORMANCE ASSESSMENT

Form 08

Engineering of analysis 3th year students

Instructor: Mahmoud Shakir Wahhab

Table 1, Plan of whole year assessments

Program Outcomes	Course Learning Objectives	Strategies for Achieving Outcomes	Assessment Method (results table after performing)
• Differential equation are basic importance in engineering mathematics because many physical laws are relations appear mathematically in the form of a differential equation • Formulate relevant research problems; conduct experimental and/or analytical study and analyzing results with modern mathematical/scientific methods and use of software tools.	• To create a congenial environment that promotes learning, growth, and imparts the ability to work with inter-disciplinary groups in professional, industry, and research organizations. • To broaden and deepen their capabilities in analytical and experimental research methods, analysis of data, and drawing relevant conclusions for scholarly writing and presentation.	1. Align goals and objectives to achieve common desire outcomes 2. Eliminate bad habits 3. Welcome Failure 4. Benefit the daily goal setting 5. Avoid procrastination	1. In-class and online quizzes 2. Homework 3. Peer feedback activities 4. Practice exams

Table 2, Assessment Rubrics

Rubric	4- Exceeds	3- Meets	2-Progressing	1-Below Average
Engineering Knowledge	Students can apply concepts of basic science and basic mathematics to solve engineering problems.	The student will just be able to understand the concepts of basic science and basic mathematics to solve engineering problems	The student will just be able to remember the concepts of basic science and basic mathematics to solve engineering problems	The student does not have an engineering sense
Problem Analysis	Student can analyze a given problem and identify the constraints and define the requirements for a given problem which are suitable for its solution	The student is just able to have a grasp of a problem statement and its constraints and can understand problem definition and the requirements for a given problem which are suitable for its solution.	Students need assistance to have a grasp of the problem statement and its constraints and can understand problem definition and the requirements for a given problem which are suitable for its solution.	The student is not able to recognize the basics of problem analysis
Design and Development of Solutions	The student can design a functional and realistic system consisting of multiple components or processes.	The student can understand and apply the engineering knowledge for the design of functional and realistic system consisting of multiple components or processes.	The student would require aid to and apply the engineering knowledge for the design functional and realistic system consisting of multiple components or processes	The student does not have the imagination to design an engineering part

Table 3, Students Works Rating

Students Outcome	Max Score
	High : 100
	Low : 50
	Mean :75
	SD : 2.5

Table 4, Student and Faculty Evaluations of Learning Outcomes

Students Outcomes	Students Rating	Instructor Rating	Instructor Comments
Not yet achieved	**Not yet achieved**	**Not yet achieved**	**Not yet achieved**

Table 5, Changes/Improvements

Assessment of Changes/Improvements Made this year	

Changes/Improvements That Will Be Made Next Time the Course is Offered	

Table 6, Final Evaluation

Outcome	Average	Notes
Not yet achieved	Not yet achieved	Not yet achieved

Appendices:

Materials: (Course notes should be here)

Faculty Curriculum Vitae:

Mahmoud Shakir Wahhab
Ass. Teacher in Electronic and control.
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University of south Ural, Russia
M, Sc. Mechatronics Engineering (2018)
Thesis: "Automation control system design of temperature stabilization of belt conveyor AC motor"

University of northern technical, Iraq
B, Sc. electronic and control (2006)

<u>Miscellaneous</u>
Computer Skills:

Matlab/Simulink/GUI
Ansys and Ansys workbench
Multisim
Solid Works
LabVIEW
Visual Basic
AUTOCAD (2D/3D)

Languages: Arabic– native language
English – Very good at reading and writing.
Russian - Good at reading and writing.
Turkish - Good at reading and writing.

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Third Stage

Mathematics

Ordinary Differential Equation (Separation of Variables)

Lecture (1)

Presented by :-
Mahmoud Shakir Wahhab

What is a differential equation?

A differential equation is an equation between specified derivative on an unknown function, its values, and known quantities and functions. Many physical laws are most simply and naturally formulated as differential equations (or DEs, as we will write for short). Ordinary differential equations are they arise most commonly in the study of dynamical systems and electrical networks. Every **first-order ordinary differential equation** can be written in the form:

$$\boldsymbol{y' = f\,(x, y)}$$

Separation of Variables

An ordinary differential equation (ODE) is an algebraic equation f (x, y, dy/dx) = 0 involving derivatives of some unknown function with respect to one independent variable. A separable ordinary differential equation is an ODE which can be written in such a way that the dependent variable and its differential appear on one side of the equals sign and the independent variable and its differential appear on the other side. A first-order ordinary differential equation is **separable** equation can be re-written in the form: $\boldsymbol{g(y)\, y' = h(x)}$

Note: the expressions involving y and y' are on one side of the equation and the expressions involving x are on the other.

Solving DEs by Separation of Variables.

The steps to solving such DEs are as follows:

- Make the DE $\frac{dy}{dx} = g(x)\,f(y)$ look like this may be already done for you (in which case you can just identify the various parts), or you may have to do some algebra to get it into the correct form.
- **Separate the variables:**

Get all the y's on the LHS by multiplying both sides by $\frac{1}{f(y)}$ or dividing by f(y)

$$\frac{1}{f\,(y)}\,\frac{dy}{dx} = g(x)$$

and get all the x's on the RHS by `multiplying' both sides by dx: $\frac{1}{f\,(y)}\,dy = g(x)\,dx$

- Integrate both sides: $\int \frac{1}{f\,(y)}\,dy = \int g(x)\,dx$.
- Solve for y (if possible). This gives us an explicit solution.
- If there is an initial condition, use it to solve for the unknown parameter in the solution function.

EXAMPLE

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2x}{y+1}$

Solution

$(y+1)\,dy = 2x\,dx \longrightarrow \int (y+1)\,dy = \int 2x\,dx$

$\int y\,dy + \int dy = \int 2x\,dx \longrightarrow \dfrac{y^2}{2} + y = x^2 + c$

EXAMPLE

Solve the differential equation $\frac{dy}{dx} = (1+x)(1+y)$

Solution

$dy = (1+x)(1+y)dx \longrightarrow [dy = (1+x)(1+y)dx\,] \quad \div (1+y)$

$\frac{1}{1+y}\,dy = (1+x)dx \longrightarrow \int \frac{1}{1+y}\,dy = \int dx + \int x\,dx$

$\ln|1+y| = x + \frac{x^2}{2} + c$

EXAMPLE

Solve the differential equation $$x\,\frac{dy}{dx} = y + xy$$

Solution

$$\frac{x\,dy}{dx} = y\,(1+x) \longrightarrow [x\,dy = y\,(1+x)\,dx] \quad \div\, xy$$

$$\frac{1}{y}\,dy = \left(\frac{1}{x}+1\right)dx \longrightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx + \int dx \longrightarrow \ln y = \ln x + x + c$$

EXAMPLE

Solve the differential equation $\dfrac{\sin x}{1+y}\dfrac{dy}{dx} = \cos x$

Solution

$$\sin x \;\, dy = (1+y)\cos x \;\, dx \longrightarrow [\sin x \;\, dy = (1+y)\cos x \;\, dx] \quad \div \sin x \;\, (1+y)$$

$$\frac{1}{1+y}\, dy = \cot x \;\, dx \longrightarrow \int \frac{1}{1+y}\, dy = \int \frac{\cos x}{\sin x}\, dx \longrightarrow \ln|1+y| = \ln|\sin x| + c$$

EXAMPLE

Solve the differential equation $e^{x+2y}\,dx + e^{3x-4y}\,dy = 0$

Solution

$$(e^{x} \cdot e^{2y})\,dx + (e^{3x} \cdot e^{-4y})\,dy = 0 \qquad * (e^{-2y} \cdot e^{-3x})$$

$$(e^{x} \cdot \cancel{e^{2y}} \cdot \cancel{e^{-2y}} \cdot e^{-3x})\,dx + (\cancel{e^{3x}} \cdot e^{-4y} \cdot e^{-2y} \cdot \cancel{e^{-3x}})\,dy = 0$$

$$e^{-2x}\,dx + e^{-6y}\,dy = 0$$

$$\int e^{-6y}\,dy = \int -e^{-2x}\,dx \quad \longrightarrow \quad \frac{-1}{6}\,e^{-6y} = \frac{1}{2}e^{-2x} + \mathrm{c}$$

EXAMPLE

Solve the differential equation $\ln y \cdot y' = \dfrac{y}{x}$

Solution

$$\frac{dy}{dx}\ln y = \frac{y}{x} \longrightarrow x\ln y \; dy = y\,dx \qquad \div\, xy$$

$$\frac{\ln y}{y}\,dy = \frac{1}{x}\,dx \longrightarrow \int \frac{\ln y}{y}\,dy = \int \frac{1}{x}\,dx \longrightarrow \frac{(\ln y)^2}{2} = \ln x + c$$

EXAMPLE

Solve the differential equation $x^2\, y\, \frac{dy}{dx} = (1 + x) \csc y$

Solution

$$x^2\, y\, dy = (1 + x) \csc y\, dx \qquad \div\ x^2 \csc y \longrightarrow \frac{y}{\csc y}\, dy = \frac{1 + x}{x^2}\, dx$$

$$\int \frac{y}{\csc y}\, dy = \int \frac{1 + x}{x^2}\, dx \longrightarrow \int y \sin y\ dy = \int \frac{1}{x^2}\, dx + \int \frac{1}{x}\, dx$$

$$-y \cos y\ + \sin y\ = \frac{-1}{x} + \ln x + c$$

$$\sin y\ - y \cos y\ = \frac{-1}{x} + \ln x + c$$

y	$+$	$\sin y$
1	$-$	$-\cos y$
0		$-\sin y$

EXAMPLE

Solve the differential equation $y' + 2x\tan y = 0$ where $y(0) = \dfrac{\pi}{2}$

Solution

$$\frac{dy}{dx} + 2x\tan y = 0 \longrightarrow \frac{dy}{dx} = -2x\tan y \longrightarrow dy = -2x\tan y\ dx \qquad \div \tan y$$

$$\frac{1}{\tan y}\,dy = -2x\ dx \longrightarrow \int \frac{\cos y}{\sin y}\,dy = \int -2x\ dx \longrightarrow \ln|\sin y| = -x^2 + c$$

$$\because y\,(0) = \frac{\pi}{2}$$

$$\ln\left|\sin\frac{\pi}{2}\right| = c$$

$$\ln|1| = c$$

$$c = 0$$

$\ln|\sin y| = -x^2$ **take e to both side** $\longrightarrow e^{\ln|\sin y|} = e^{-x^2}$

$$\sin y = e^{-x^2} \quad * \ \sin^{-1} \longrightarrow y = \sin^{-1}(e^{-x^2})$$

EXAMPLE

Solve the differential equation $y' = x^2 y^2 - 4x^2$

Solution

$$\frac{dy}{dx} = x^2(y^2 - 4) \longrightarrow dy = x^2(y^2 - 4)\,dx \qquad \div\ (y^2 - 4)$$

$$\frac{1}{y^2 - 4}\,dy = x^2\,dx \longrightarrow \int \frac{1}{y^2 - 4}\,dy = \int x^2\,dx$$

$$\frac{1}{y^2 - 4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$A = \frac{1}{\cancel{(y-2)}(y+2)} \cdot \cancel{(y-2)}\Bigg|_{y=2} \longrightarrow A = \frac{1}{4}$$

$$B = \frac{1}{(y-2)\cancel{(y+2)}} \cdot \cancel{(y+2)} \Bigg|_{y=-2} \longrightarrow B = \frac{-1}{4}$$

$$\frac{1}{4}\int \frac{1}{y-2}\,dy - \frac{1}{4}\int \frac{1}{y+2}\,dy = \int x^2\,dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = \frac{1}{3}x^3 + c$$

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = \frac{1}{3}x^3 + c \quad *3 \longrightarrow \frac{3}{4}\ln\left|\frac{y-2}{y+2}\right| = x^3 + 3c$$

take e to both side

$$\left(\frac{y-2}{y+2}\right)^{3/4} = e^{x^3+k}$$

Where k = 3c

Thank you
for listening
any question

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Third Stage

Mathematics

Homogeneous Differential Equations

Lecture (2)

Presented by :-
Mahmoud Shakir Wahhab

Theory

$M(x,y) = 3x^2 + xy$ is a homogeneous function since the sum of the powers of x and y in each term is the same (i.e. x^2 is x to power 2 and $xy = x^1y^1$ giving total power of 1 + 1 = 2). The degree of this homogeneous function is 2.

Here, we consider differential equations with the following standard form:

$$\frac{dy}{dx} = \frac{M\,(x,y)}{N\,(x,y)}$$

where M and N are homogeneous functions of the same degree.

How to Solve Homogeneous Differential Equations

- To find the solution, change the dependent variable from y to v, where $y = vx$
- Using the product rule for differentiation.
- The LHS of the equation becomes: $\frac{dy}{dx} = v + x\frac{dv}{dx}$
- Solve the resulting equation by separating the variables v and x.
- Finally, re-express the solution in terms of x and y

<u>Note.</u> This method also works for equations of the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

EXAMPLE

Solve the differential equation $\frac{dy}{dx}=\frac{x+3y}{2x}$

$$let\ v=\frac{y}{x}\Rightarrow y=vx \longrightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{x+3(vx)}{2x}\Rightarrow v+x\frac{dv}{dx}=\frac{x+3vx}{2x}$$

$$v+x\frac{dv}{dx}=\frac{1+3v}{2}\Rightarrow x\frac{dv}{dx}=\frac{1+3v}{2}-v$$

$$x\frac{dv}{dx} = \frac{1+v}{2} \longrightarrow 2x\,dv = (1+v)\,dx \qquad \div\, x\,(1+v)$$

$$\frac{2}{(1+v)}\,dv = \frac{1}{x}dx \longrightarrow 2\int \frac{1}{(1+v)}\,dv = \int \frac{1}{x}dx$$

$$2\ \ln|1+v| = \ln|x| + c$$

$$2\ \ln\left|1+\frac{y}{x}\right| = \ln|x| + c$$

EXAMPLE

Solve the differential equation $(x^2 + xy)\frac{dy}{dx} = xy - y^2$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2 + xy}$$

$$let\ v = \frac{y}{x} \Rrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{x(vx) - (vx)^2}{x^2 + x(vx)} \Rrightarrow \frac{dy}{dx} = \frac{vx^2 - v^2x^2}{x^2 + vx^2}$$

$$\frac{dy}{dx} = \frac{v - v^2}{1 + v}$$

$$v + x\frac{dv}{dx} = \frac{v - v^2}{1 + v} \Rrightarrow x\frac{dv}{dx} = \frac{-2v^2}{1 + v} \longrightarrow x(1 + v)dv = -2v^2dx \quad \div xv^2$$

$$\frac{(1 + v)}{v^2}dv = \frac{-2}{x}dx \Rrightarrow \left(\frac{1}{v^2} + \frac{1}{v}\right)dv = \frac{-2}{x}dx$$

$$\int v^{-2}\,dv + \int \frac{1}{v}\,dv = -2\int \frac{1}{x}\,dx \longrightarrow \frac{-1}{v} + \ln|v| = -2\ln|x| + c$$

$$\frac{-x}{y} + \ln\left|\frac{y}{x}\right| = -2\ln|x| + c$$

EXAMPLE

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}\,(1 + \ln y - \ln x)$

$$\frac{dy}{dx} = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right)$$

$$let\ v = \frac{y}{x} \Rrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v\,(1 + \ln v)$$

$$v + x\frac{dv}{dx} = v + v\ln v$$

$$x\frac{dv}{dx} = v\ln v \longrightarrow x\,dv = v\ln v\;dx$$

$$\frac{1}{v\ln v}\,dv = \frac{1}{x}\,dx \Rightarrow \int \frac{1}{v\ln v}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|\ln|v|| = \ln|x| + c$$

$$\ln\left|\ln\left|\frac{y}{x}\right|\right| = \ln|x| + c$$

EXAMPLE

Solve the differential equation $(xy + x^2)dy + y^2\,dx = 0$

$$\frac{dy}{dx} = \frac{-y^2}{xy + x^2}$$

$$let\ v = \frac{y}{x} \Rrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-y^2}{xy + x^2} \Rrightarrow v + x\frac{dv}{dx} = \frac{-(vx)^2}{x(vx) + x^2}$$

$$v + x\frac{dv}{dx} = \frac{-v^2}{v+1} \Rrightarrow x\frac{dv}{dx} = \frac{-v^2}{v+1} - v$$

$$x\frac{dv}{dx} = \frac{-v^2 - v(v+1)}{(v+1)} \Rrightarrow x\frac{dv}{dx} = \frac{-2v^2 - v}{v+1}$$

$$\frac{-v-1}{v(2v+1)}\,dv = \frac{1}{x}\,dx \Rrightarrow \int \frac{-v-1}{v(2v+1)}\,dv = \int \frac{1}{x}\,dx$$

To solve the integral, we use the partial fraction method

$$\frac{-v-1}{v(2v+1)} = \frac{A}{v} + \frac{B}{2v+1}$$

$$A = \frac{-v-1}{v(2v+1)} \cdot v \,\Bigg|_{v=0} \Rrightarrow A = -1$$

$$B = \frac{-v-1}{v(2v+1)} \cdot (2v+1) \bigg|_{v=\frac{-1}{2}} \qquad \Rightarrow B = 1$$

$$\int \frac{-1}{v}\, dv + \int \frac{1}{2v+1}\, dv \cdot \frac{2}{2} = \int \frac{1}{x}\, dx$$

$$-\ln|v| + \frac{1}{2}\ln|2v+1| = \ln|x| + c \;\Rightarrow -\ln\left|\frac{y}{x}\right| + \frac{1}{2}\ln\left|\frac{2y}{x}+1\right| = \ln|x| + c$$

$$-\ln|y| + \ln x + \frac{1}{2}\ln\left|\frac{2y}{x}+1\right| - \ln|x| = c \;\Rightarrow \ln|y| - \frac{1}{2}\ln\left|\frac{2y}{x}+1\right| = -c$$

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Third Stage

Mathematics

Linear and Bernoulli Differential Equations

Lecture (3)

Presented by :-
Mahmoud Shakir Wahhab

Linear equations

- Consider an ordinary differential equation that we wish to solve to find out how the variable y depends on the variable x.
- Any such linear first order O.D.E. can be re-arranged to give the following standard form:

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

- Where $P(x)$ and $Q(x)$ are functions of x, and in some cases may be constants.

- A linear first order O.D.E. can be solved using the integrating factor method.

- After writing the equation in standard form, P(x) can be identified.
- One then multiplies the equation by the following “integrating factor”:

$$IF = e^{\int P(x)\,dx}$$

- This factor is defined so that the equation becomes equivalent to:

$$\frac{d}{dx}(IF\,y) = IF\,Q(x)$$

- Where by integrating both sides with respect to x, gives:

$$y \cdot IF = \int IF\,Q(x)\,dx$$

- Finally, division by the integrating factor (IF) gives y explicitly in terms of x, i.e. gives the solution to the equation.

EXAMPLE

Solve the differential equation $\frac{dy}{dx} + 2y = e^{-x}$

Solution:- $P(x) = 2 \ , Q(x) = e^{-x}$

$$IF = e^{\int p(x)\,dx} \Rightarrow IF = e^{\int 2dx} \Rightarrow IF = e^{2x}$$

$$y \cdot IF = \int IF \cdot Q(x)\,dx$$

$$y \cdot e^{2x} = \int e^{2x} \cdot e^{-x}\,dx \Rightarrow ye^{2x} = \int e^{x}\,dx$$

$$y\,e^{2x} = e^{x} + c \longrightarrow y = \frac{e^{x} + c}{e^{2x}}$$

EXAMPLE

Solve the differential equation $x\,\frac{dy}{dx} - 3y = x^2$

Solution:-

$$\frac{dy}{dx} - \frac{3}{x}y = x \longrightarrow p(x) = \frac{-3}{x} \quad , \quad \mathrm{Q(x) = x}$$

$$IF = e^{\int p(x)dx} = e^{\int \frac{-3}{x}dx} \longrightarrow e^{-3\ln x} = e^{\ln x^{-3}} \longrightarrow IF = \frac{1}{x^3}$$

$$y \cdot IF = \int IF \cdot Q(x)dx \longrightarrow y \cdot \frac{1}{x^3} = \int \frac{1}{x^3} \cdot x\,dx \longrightarrow \frac{y}{x^3} = \int x^{-2}\,dx$$

$$\frac{y}{x^3} = \frac{x^{-1}}{-1} + c \longrightarrow \frac{y}{x^3} = \frac{-1}{x} + c \longrightarrow y = x^3\left(\frac{-1}{x} + c\right)$$

EXAMPLE

Solve the differential equation $y\dfrac{dx}{dy} + x = \sin y$

Solution:-

$$\frac{dx}{dy} + \frac{1}{y}x = \frac{\sin y}{y} \longrightarrow p(y) = \frac{1}{y} \quad , Q(y) = \frac{\sin y}{y}$$

$$IF = e^{\int p(y)dy} = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$$

$$x \cdot IF = \int IF \cdot Q(y)\, dy \longrightarrow x \cdot y = \int y \cdot \frac{\sin y}{y}\, dy$$

$$x \cdot y = -\cos y + c \longrightarrow x = \frac{-\cos y + c}{y}$$

Bernoulli equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called ***Bernoulli equation.***

$$y' + P(x)\, y = Q(x)\, y^n$$

This equation looks a lot like a linear equation except for the y^n. If n=0 or n=1, then the equation is linear and we can solve it. Otherwise, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

EXAMPLE

Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = xy^2$

Solution:-

Multiply the equation by y^{-2} $\longrightarrow$ $y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x$

$let\ u = y^{-1}$ $\longrightarrow$ $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$ Multiply both sides by -1

$-\frac{du}{dx} = y^{-2}\frac{dy}{dx}$ $\longrightarrow$ $-\frac{du}{dx} + \frac{1}{x}u = x$ Multiply both side by -1

$\frac{du}{dx} - \frac{1}{x}u = -x$ $\longrightarrow$ $p(x) = \frac{-1}{x}$, $\mathrm{Q(x)} = -\mathrm{x}$

$$IF = e^{\int p(x)dx} = e^{\int \frac{-1}{x} dx} \longrightarrow e^{-\ln x} = e^{\ln x^{-1}} \longrightarrow IF = \frac{1}{x}$$

$$u \cdot IF = \int IF \cdot Q(x)dx$$

$$u \cdot \frac{1}{x} = \int \frac{1}{x} \cdot (-x)\, dx \longrightarrow \frac{u}{x} = \int -\, dx$$

$$\frac{u}{x} = -x + c \longrightarrow u = x(-x + c)$$

$$\frac{1}{y} = -x^2 + cx \longrightarrow y = \frac{1}{cx - x^2}$$

EXAMPLE

Solve the differential equation $\frac{dy}{\sqrt{y}dx} + \sqrt{y} = \frac{1}{y}$

Solution:-

Multiply both sides by $\sqrt{y}$ we get $\frac{dy}{dx} + y = \frac{1}{\sqrt{y}}$

$\frac{dy}{dx} + y = y^{\frac{-1}{2}}$ Multiply both sides by $y^{\frac{1}{2}}$

$$y^{\frac{1}{2}}\frac{dy}{dx} + y^{\frac{3}{2}} = 1$$

$let\ u = y^{\frac{3}{2}} \longrightarrow \frac{du}{dx} = \frac{3}{2}y^{\frac{1}{2}}\frac{dy}{dx} \longrightarrow \frac{2}{3}\frac{du}{dx} = y^{\frac{1}{2}}\frac{dy}{dx}$

$$\frac{2}{3}\frac{du}{dx} + u = 1$$

Multiply both sides by $\frac{3}{2}$

$$\frac{du}{dx} + \frac{3}{2}u = \frac{3}{2} \longrightarrow P(x) = \frac{3}{2} \quad , \quad \mathrm{Q(x)} = \frac{3}{2}$$

$$IF = e^{\int P(x)dx} \longrightarrow e^{\int \frac{3}{2} dx} \Rrightarrow e^{\frac{3}{2}x}$$

$$u \cdot IF = \int IF \cdot Q(x)\, dx \longrightarrow u \cdot e^{\frac{3}{2}x} = \int e^{\frac{3}{2}x} \cdot \frac{3}{2}\, dx$$

$$u \cdot e^{\frac{3}{2}x} = e^{\frac{3}{2}x} + c \longrightarrow y^{\frac{3}{2}} \cdot e^{\frac{3}{2}x} = e^{\frac{3}{2}x} + \mathrm{c}$$

$$y^{\frac{3}{2}} = \frac{e^{\frac{3}{2}x} + \mathrm{c}}{e^{\frac{3}{2}x}}$$

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Third Stage

Mathematics

Exact and NonExact Differential Equations

Lecture (4)

Presented by :-
Mahmoud Shakir Wahhab

Exact Differential Equations

Consider the differential equation

$$\mathrm{F} = M\,(x, y)\,dx + N\,(x, y)dy = 0$$

where M and N are both continuously differentiable functions with continuous partials $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$. This equation will be called **Exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

How to Solve Exact Differential Equation

The following steps explains how to solve the exact differential equation in a detailed way.

Step 1: The first step to solve exact differential equation is that to make sure with the given differential equation is exact using testing for exactness.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 2: Integrate either the first equation with respect of the variable x or the second with respect of the variable y. The choice of the equation to be integrated will depend on how easy the calculations are. Let us assume that the first equation was chosen, then we get

$$\boldsymbol{F\,(x,y) = \int M\,(x,y)dx + g(y)}$$

The function $g(y)$ should be there, since in our integration, we assumed that the variable y is constant.

Or assume that the second equation was chosen, then we get

$$\boldsymbol{F\,(x,y) = \int N\,(x,y)dy + g(x)}$$

The function $g(x)$ should be there, since in our integration, we assumed that the variable x is constant.

Step 3:- Differentiating with respect to y,

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M\,(x,y)dx + g(y)\right] = N(x,y)$$

From the above expression we get the derivative of the unknown function $g(y)$ and it is given by

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M\,(x,y)dx\right]$$

Or differentiating with respect to x,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[\int N\,(x, y)dy + g(x)\right] = M(x, y)$$

From the above expression we get the derivative of the unknown function $g(x)$ and it is given by

$$g'(x) = M(x, y) - \frac{\partial}{\partial x}\left[\int N\,(x, y)dy\right]$$

Step 4:- We can find the function $g'(y)$ by integrating the last expression. Or we can find the function $g'(x)$ by integrating the last expression.

Step 5:- Finally, the general solution of the exact differential equation is given by

$$F(x,y) = \int M\,(x,y)dx + \int g'(y)dy$$

OR

$$F(x,y) = \int N\,(x,y)dy + \int g'(x)dx$$

EXAMPLE

Solve the differential equation $2xy\,dx + (x^2 + \cos y)\,dy = 0$

Solution

$$\left.\begin{aligned} let\ M = 2xy \quad &\Rightarrow \frac{\partial M}{\partial y} = 2x \\ let\ N = x^2 + \cos y \quad &\Rightarrow \frac{\partial N}{\partial x} = 2x \end{aligned}\right\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{The D.E is Exact.}$$

$$F = \int M\,(x, y)dx + g(y) \longrightarrow F = \int 2xy\ dx + g(y) = x^2y + g(y)$$

$$F = (x^2y + g(y)) \longrightarrow \frac{\partial F}{\partial y} = x^2 + g'(y) = N(x,y)$$

$$g'(y) = x^2 + \cos y - x^2 \longrightarrow g'(y) = \cos \mathrm{y}$$

$$\int g'(y)dy = \int \cos \mathrm{y}\, dy \longrightarrow \sin y + k$$

$$F(x,y) = x^2y + \sin y + k$$

EXAMPLE

Solve the differential equation $(x^2+y^2)\,dx + (2xy + \cos y)dy = 0$

Solution

$$\left.\begin{aligned} &let\ M = (x^2+y^2) \Rrightarrow \frac{\partial M}{\partial y} = 2y \\ &let\ N = 2xy + \cos y \Rrightarrow \frac{\partial N}{\partial x} = 2y \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{The D.E is Exact.}$$

$$F = \int N\,(x,y)dy + g(x) \longrightarrow F = \int (2xy + \cos y)\ dy + g(x) = xy^2 + \sin y + g(x)$$

$$F = (xy^2 + \sin y + g(x)) \longrightarrow \frac{\partial F}{\partial x} = y^2 + g'(x) = M(x, y)$$

$$g'(x) = x^2 + y^2 - y^2 \longrightarrow g'(x) = x^2$$

$$\int g'(x)dx = \int x^2\,dx \longrightarrow g(x) = \frac{x^3}{3} + k$$

$$F\,(x, y) = xy^2 + \sin y + \frac{x^3}{3} + k$$

Integrating Factors

Sometimes a differential equation $M\ (x, y)\ dx + N\ (x, y)dy = 0$ is not exact,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

but can be made exact by multiplying by an integrating factor.

Integrating Factors

To determination of the integrating factor, there are two cases

Case 1: There exists an integrating factor $\boldsymbol{\rho}\ (\boldsymbol{x})$ function of x only. This happens if the expression

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$$

Is a function of x only, that is the variable y disappears from the expression. In this case, the function $\boldsymbol{\rho}$ is given by

$$\rho(x)=\exp\left(\int\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}\,dx\right)$$

Case 2: There exists an integrating factor $\boldsymbol{\rho}\,(\boldsymbol{y})$ function of y only. This happens if the expression

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

Is a function of y only, that is the variable x disappears from the expression. In this case, the function $\boldsymbol{\rho}$ is given by

$$\rho(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy \right)$$

EXAMPLE

Solve the differential equation $(x+2y)dx - x\,dy = 0$

Solution

$$let\ M = (x+2y) \Rrightarrow \frac{\partial M}{\partial y} = 2.$$

$$let\ N = -x \Rrightarrow \frac{\partial N}{\partial x} = -1$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ The D. E is not Exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \longrightarrow \frac{2+1}{-x} = \frac{-3}{x}$$

$$\rho(x) = e^{\int \frac{-3}{x} dx} \longrightarrow e^{-3 \ln x} \longrightarrow \rho(x) = \frac{1}{x^3}$$

Once the integrating factor is found, multiply the old equation by $\rho(x)$ to get a new one which is exact.

$$(x + 2y) \cdot \frac{1}{x^3}\, dx - x \cdot \frac{1}{x^3} dy = 0 \longrightarrow \frac{(x + 2y)}{x^3}\, dx - \frac{1}{x^2} dy = 0$$

Which is exact. (Check it!)

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{2}{x^3} \\ \frac{\partial N}{\partial x} &= \frac{2}{x^3} \end{aligned}\right\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ The D.E is Exact.}$$

$$F = \int N(x,y)dy + g(x) \longrightarrow F = \int -\frac{1}{x^2}\, dy + g(x) = \frac{-y}{x^2} + g(x)$$

$$F = \frac{-y}{x^2} + g(x) \longrightarrow \frac{\partial F}{\partial x} = \frac{2y}{x^3} + g'(x) = M(x,y)$$

$$g'(x) = \frac{(x+2y)}{x^3} - \frac{2y}{x^3} \longrightarrow g'(x) = \frac{1}{x^2}$$

$$\int g'(x)\, dx \Rightarrow \int \frac{1}{x^2}\, dx \longrightarrow g(x) = \frac{-1}{x} + k$$

$$F(x,y) = \frac{-y}{x^2} - \frac{1}{x} + k$$

Thank you for listening any question

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Mathematics

Ordinary Differential Equation

Lecture (5)

Presented by :-
Mahmoud Shakir Wahhab

EXAMPLE

Solve the differential equation $dy + xydx - 3xdx = 0$

Solution

$\frac{dy}{dx} + xy - 3x = 0 \longrightarrow \frac{dy}{dx} + xy = 3x \longrightarrow$ Linear differential equation

$P(x) = x \ , Q(x) = 3x$

$IF = e^{\int P(x)dx} \longrightarrow IF = e^{\int xdx} \longrightarrow IF = e^{\frac{1}{2}x^2}$

$y \cdot IF = \int IF \cdot Q(x)dx \longrightarrow y \cdot e^{\frac{1}{2}x^2} = \int e^{\frac{1}{2}x^2} \cdot 3xdx$

$y \cdot e^{\frac{1}{2}x^2} = 3e^{\frac{1}{2}x^2} + \mathrm{c} \longrightarrow y = \frac{3e^{\frac{1}{2}x^2} + \mathrm{c}}{e^{\frac{1}{2}x^2}}$

EXAMPLE

Solve the differential equation $dy + xydx - 3xdx = 0$

Solution

$dy + (xy - 3x)dx = 0 \longrightarrow \mathrm{d}y = (3x - xy)dx \longrightarrow \mathrm{d}y = x(3 - y)dx$

$\frac{1}{(3-y)}\, dy = x\, dx \longrightarrow \int \frac{1}{(3-y)}\, dy = \int x\, dx \longrightarrow -\int \frac{-1}{(3-y)}\, dy = \int x\, dx$

$-\ln(3 - y) = \frac{1}{2}x^2 + c \longrightarrow \ln(3 - y)^{-1} = \frac{1}{2}x^2 + c \longrightarrow \frac{1}{3-y} = e^{\frac{1}{2}x^2 + c}$

$1 = 3e^{\frac{1}{2}x^2+c} - ye^{\frac{1}{2}x^2+c} \longrightarrow y = \frac{3e^{\frac{1}{2}x^2+c} - 1}{e^{\frac{1}{2}x^2+c}}$

EXAMPLE

Solve the differential equation $xdx + 2xdy = -3ydy - 2ydx$

Solution

$$xdx + 2ydx + 2xdy + 3ydy = 0 \longrightarrow (x + 2y)dx + (2x + 3y)dy = 0$$

$$\left.\begin{aligned} let\ M = x + 2y \Rrightarrow \frac{\partial M}{\partial y} = 2 \\ let\ N = 2x + 3y \Rrightarrow \frac{\partial N}{\partial x} = 2 \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{The D.E is Exact.}$$

$$F = \int M\,(x, y)dx + g(y) \longrightarrow F = \int (x + 2y)\ dx + g(y) = \frac{1}{2}x^2 + 2xy + g(y)$$

$$F = \frac{1}{2}x^2 + 2xy + g(y) \longrightarrow \frac{\partial F}{\partial y} = 2x + g'(y) = N(x, y)$$

$$g'(y) = 2x + 3y - 2x \longrightarrow g'(y) = 3y$$

$$\int g'(y)dy = \int 3y\, dy \longrightarrow \frac{3}{2}y^2 + k$$

$$F = \frac{1}{2}x^2 + 2xy + \frac{3}{2}y^2 + k$$

EXAMPLE

Solve the differential equation $xdx + 2xdy = -3ydy - 2ydx$

Solution

$$2xdy + 3ydy = -xdx - 2ydx \longrightarrow (2x + 3y)dy = -(x + 2y)dx$$

$$\frac{dy}{dx} = \frac{-(x + 2y)}{(2x + 3y)}$$

$$let\ v = \frac{y}{x} \Rrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-(x + 2vx)}{(2x + 3vx)} \longrightarrow v + x\frac{dv}{dx} = \frac{-(1 + 2v)}{(2 + 3v)}$$

$$v + x\frac{dv}{dx} = \frac{-(1+2v)}{(2+3v)} \longrightarrow x\frac{dv}{dx} = \frac{-(1+2v)}{(2+3v)} - v$$

$$x\frac{dv}{dx} = \frac{-(1+2v)}{(2+3v)} - \frac{v(2+3v)}{(2+3v)} \longrightarrow x\frac{dv}{dx} = \frac{-1-2v-2v-3v^2}{(2+3v)}$$

$$x\frac{dv}{dx} = \frac{-(3v^2+4v+1)}{(2+3v)} \longrightarrow \frac{(2+3v)}{(3v^2+4v+1)}\,dv = \frac{-1}{x}\,dx$$

$$\int \frac{(2+3v)}{(3v^2+4v+1)}\,dv = \int \frac{-1}{x}\,dx \longrightarrow \frac{1}{2}\int \frac{2(2+3v)}{(3v^2+4v+1)}\,dv = \int \frac{-1}{x}\,dx$$

$$\frac{1}{2}\ln(3v^2+4v+1) = -\ln x + c \longrightarrow \frac{1}{2}\ln(3\frac{y^2}{x^2} + 4\frac{y}{x} + 1) = -\ln x + c$$

$$\frac{1}{2}\ln(3\frac{y^2}{x^2}+4\frac{y}{x}+1)=-\ln x+c \longrightarrow \frac{1}{2}\ln\left(\frac{3y^2+4xy+x^2}{x^2}\right)=-\ln x+c$$

$$[\ln(3y^2+4xy+x^2)-\ln x^2]=-2\ln x+2c$$

$$\ln(3y^2+4xy+x^2)-2\ln x=-2\ln x+2c$$

$$\ln(3y^2+4xy+x^2)=-2\ln x+2c+2\ln x$$

$$\ln(3y^2+4xy+x^2)=2c \longrightarrow 3y^2+4xy+x^2=e^{2c}$$

EXAMPLE

Solve the differential equation $\frac{dy}{dx} = y^4 \sin x - y \cot x$

Solution

$$\frac{dy}{dx} + y \cot x = y^4 \sin x \longrightarrow \times\ y^{-4}$$

$$y^{-4}\frac{dy}{dx} + y^{-3} \cot x = \sin x \dashrightarrow 1$$

$$let\ u = y^{-3} \longrightarrow \frac{du}{dx} = -3y^{-4}\ \frac{dy}{dx}$$

Multiply both sides by $\frac{-1}{3}$

$$\frac{-1}{3}\frac{du}{dx} = y^{-4}\ \frac{dy}{dx} \dashrightarrow 2$$

Sub. 2 in 1

$$\frac{-1}{3}\ \frac{du}{dx} + u \cot x = \sin x \qquad \times \quad -3$$

$$\frac{du}{dx} - 3u\cot x = -3\sin x$$

$$P(x) = -3\cot x \qquad , \qquad \mathrm{Q(x)} = -3\sin x$$

$$IF = e^{\int P(x)dx} = e^{\int -3\cot x\, dx} = e^{-3\int \frac{\cos x}{\sin x} dx} = e^{-3\ln\sin x} = \frac{1}{\sin^3 x}$$

$$u \cdot IF = \int IF \cdot Q(x)\, dx \longrightarrow u \cdot \frac{1}{\sin^3 x} = \int \frac{1}{\sin^3 x} \cdot -3\sin x\ dx$$

$$\frac{u}{\sin^3 x} = -3\int \frac{1}{\sin^2 x} dx \longrightarrow \frac{u}{\sin^3 x} = -3\int \csc^2 x\ dx$$

$$\frac{u}{\sin^3 x} = 3\cot x + c \longrightarrow u = \sin^3 x\,(3\cot x + c)$$

$$\frac{1}{y^3} = \sin^3 x\,(3\cot x + c) \longrightarrow y^3 = \frac{1}{\sin^3 x\,(3\cot x + c)}$$

EXAMPLE

Solve the differential equation $(x^2 + y^2 + 2x)dx + 2y\,dy = 0$

Solution

$$\left.\begin{array}{l} let\ M = x^2 + y^2 + 2x \Rrightarrow \dfrac{\partial M}{\partial y} = 2y \\[2ex] let\ N = 2y \Rrightarrow \dfrac{\partial N}{\partial x} = 0 \end{array}\right\} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The D. E is not Exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \longrightarrow \frac{2y - 0}{2y} = 1$$

$$\rho(x) = e^{\int dx} \longrightarrow e^x$$

$$e^x(x^2 + y^2 + 2x)dx + e^x \cdot 2ydy = 0$$

$$let\ M = e^x(x^2 + y^2 + 2x) \Rrightarrow \frac{\partial M}{\partial y} = 2ye^x$$

$$let\ N = e^x 2y \Rrightarrow \frac{\partial N}{\partial x} = 2ye^x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ The D. E is Exact.

$$F = \int N(x,y)dy + g(x) \longrightarrow F = \int 2ye^x\, dy + g(x)$$

$$F = y^2e^x + g(x) \longrightarrow \frac{\partial F}{\partial x} = y^2e^x + g'(x)$$

$$\int g'(x)\, dx \longrightarrow \int (x^2e^x + 2xe^x)dx$$

$$x^2e^x - 2xe^x + 2e^x + 2xe^x - 2e^x + c \longrightarrow g(x) = x^2e^x + \mathrm{c}$$

$$F = y^2e^x + x^2e^x + \mathrm{c}$$

x^2		e^x
	$+$	
$2x$		e^x
	$-$	
2		e^x
	$+$	
0		e^x

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Mathematics

Second Order Differential Equations (Homogeneous)

Lecture (6)

Presented by :-
Mahmoud Shakir Wahhab

Second Order Differential Equations

Second order differential equations simply have a second derivative of the dependent variable. A second order differential equation which can be put in the form:

$$P(x)\,\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = G(x)$$

Where P(x); Q(x); R(x) and G(x) are continuous functions of x on a given interval.

If $G(x) = 0$ then the equation is called **homogeneous**. Otherwise is called **nonhomogeneous (or inhomogeneous).**

Constant coefficient second order linear ODEs

The general second order homogeneous linear differential equation with constant coefficients is

$$Ay'' + By' + Cy = 0$$

where y is an unknown function of the variable x, and A, B, and C are constants. If A = 0 this becomes a first order linear equation, which in this case is separable, and so we already know how to solve. So we will consider the case $A \neq 0$.

Such equations are used widely in the modelling of physical phenomena, for example, in the analysis of vibrating systems and the analysis of electrical circuits.

Such an equation arises for the charge on a capacitor in an unpowered RLC electrical circuit, or for the position of a freely-oscillating frictional mass on a spring, or for a damped pendulum.

The characteristic equation or auxiliary equation $ar^2 + br + c = 0$. So if we can find two values of r satisfying this much easier quadratic equation, by using the quadratic formula $r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

There are three kinds of behavior to the values of r we get, based on the discriminant $D = b^2 - 4ac$ of the quadratic:

Case1:- $D > 0$. In this case, we get the two different real numbers $r_1 = \frac{-b+\sqrt{D}}{2a}$ and $r_2 = \frac{-b-\sqrt{D}}{2a}$, and the general solution is $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

Case2:- $D = 0$. In this case both roots are equal, so we only get one value $r = \frac{-b}{2a}$, therefore we have a general solution of $y = C_1 e^{rx} + C_2 x e^{rx}$

Case3:- $D < 0$. In this case we get two complex conjugate values of r, namely

$r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, and the general solution is

$y = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x)$

EXAMPLE

Find the general solution of $y'' + 2y' - 8y = 0$

Solution

Auxiliary Equation: $r^2 + 2r - 8 = 0$

a= 1;
b= 2;
c= -8

$$r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \longrightarrow r_{1,2} = \frac{-2 \mp \sqrt{(2)^2 - 4(1)(-8)}}{2(1)}$$

$r_1 = 2 \;\; , r_2 = -4$ $\longrightarrow$ we got the two different real numbers

The general solution is $y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y = C_1 e^{2x} + C_2 e^{-4x}$

EXAMPLE

Find the general solution of $y'' - 2y' + 5y = 0$

Solution

Auxiliary Equation: $r^2 - 2r + 5 = 0$

a= 1;
b= -2;
c= 5;

$$r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \longrightarrow r_{1,2} = \frac{-(-2) \mp \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$r_{1,2} = \frac{2 \mp \sqrt{-16}}{2} \longrightarrow r_{1,2} = 1 \mp 2i \longrightarrow r_1 = 1 + 2i \quad , r_2 = 1 - 2i$$

we got two complex conjugate values of r

$$y = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x) \longrightarrow y = e^{1x}(C_1 \sin 2x + C_2 \cos 2x)$$

Initial Value Problems (IVP)

An Initial Value Problem for second-order differential equations asks for a specific the solution to the differential equation that also satisfies two initial conditions of the form:

$$y\,(x_0) = y_0\ , \qquad y'\,(x_0) = \ y_1$$

EXAMPLE

Find the initial value problem $2y'' + 3y' + y = 0$ $\quad y(0) = 1\,, \quad y'(0) = 2$

Solution

a= 2;
b= 3;
c= 1;

Auxiliary Equation: $2r^2 + 3r + 1 = 0$

$$r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \longrightarrow r_{1,2} = \frac{-3 \mp \sqrt{(3)^2 - 4(2)(1)}}{2(2)}$$

$$r_{1,2} = \frac{-3 \mp 1}{4} \longrightarrow r_1 = -1 \;, r_2 = \frac{-1}{2} \longrightarrow$$ we got the two different real numbers

The general solution is $y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y = C_1 e^{-x} + C_2 e^{\frac{-1}{2}x}$

Initial Value : now we plug in to find the constants C_1 *and* C_2 :

$$y(x) = C_1 e^{-x} + C_2 e^{\frac{-1}{2}x}$$

$$y(0) = C_1 e^{0} + C_2 e^{\frac{-1}{2}\times 0} \longrightarrow 1 = C_1 + C_2$$

$$C_1 + C_2 = 1 \dashrightarrow 1$$

$$y'(x) = -C_1 e^{-x} - \frac{1}{2} C_2 e^{\frac{-1}{2}x}$$

$$y'(0) = -C_1 e^{0} - \frac{1}{2} C_2 e^{\frac{-1}{2}\times 0} \longrightarrow 2 = -C_1 - \frac{1}{2} C_2$$

$$-C_1 - \frac{1}{2} C_2 = 2 \dashrightarrow 2$$

$$\cancel{C_1} + C_2 = 1$$

$$\cancel{-C_1} - \frac{1}{2}C_2 = 2$$

$$\frac{1}{2}C_2 = 3 \longrightarrow C_2 = 6$$

Substituting value of C_2 into the first equation, we get $C_1 = -5$

The solution to the Initial Value problem is

$$y = -5\, e^{-x} + 6\, e^{\frac{-1}{2}x}$$

EXAMPLE

Find the initial value problem $y'' + 9y = 0$ $\qquad y\left(\frac{\pi}{6}\right) = 2\,, \qquad y'\left(\frac{\pi}{6}\right) = 3$

Solution

a= 1;
b= 0;
c= 9;

Auxiliary Equation: $r^2 + 9 = 0$

$$r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \longrightarrow r_{1,2} = \frac{0 \mp \sqrt{(0)^2 - 4(1)(9)}}{2(1)}$$

$$r_{1,2} = \frac{\sqrt{-36}}{2} \longrightarrow r_{1,2} = \frac{\sqrt{-1}\,\sqrt{36}}{2} \longrightarrow r_{1,2} = \mp 3\,i$$

The general solution is $y = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x)$

$$y = e^{0x}(C_1 \sin 3x + C_2 \cos 3x) \longrightarrow y = (C_1 \sin 3x + C_2 \cos 3x)$$

Initial Value : now we plug in to find the constants C_1 and C_2 :

$$y\left(\frac{\pi}{6}\right) = \left\{C_1 \sin(3 \times \frac{\pi}{6}) + C_2 \cos(3 \times \frac{\pi}{6})\right\}$$

$$2 = (C_1 \times 1 + C_2 \times 0) \longrightarrow C_1 = 2$$

$$y = (C_1 \sin 3x + C_2 \cos 3x)$$

$$y' = (3C_1 \cos 3x - 3C_2 \sin 3x)$$

$$y'\left(\frac{\pi}{6}\right) = \left(3 \times 2 \cos 3 \times \frac{\pi}{6} - 3C_2 \sin 3 \times \frac{\pi}{6}\right)$$

$$3 = -3C_2 \longrightarrow C_2 = -1$$

The solution to the Initial Value problem is $\longrightarrow$ $y = (2 \sin 3x - \cos 3x)$

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ANY QUESTION

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Mathematics

Second Order Differential Equations (Nonhomogeneous) part I

Lecture (7)

Presented by :-
Mahmoud Shakir Wahhab

Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients

In the previous lecture focused on finding the general solution of homogeneous linear constant-coefficient second-order differential equations.

$$Ay'' + By' + Cy = 0$$

- In this lecture explains how to find the general solution of **nonhomogeneous** linear constant-coefficient second-order differential equations–that is, equations of the form:

$$Ay'' + By' + Cy = f(x)$$

where A, B, C are constants, with, $A \neq 0$, and f(x) is a given continuous function of x and $f(x) \neq 0$. The basic method for finding the general solution of such an equation depends on the principle of superposition.

The method of undetermined coefficients

The Method of Undetermined Coefficients is just a fancy way of making an educated guess about what the form of the solution will be and then checking if it works. The general solution $y_{G.S}(x) = y_c(x) + y_p(x)$

The term $y_c(x) = C_1\, y_1 + C_2\, y_2$ is called the *complementary solution* (or the *homogeneous solution*) of the nonhomogeneous equation. The term $y_p(x)$ is called the *particular solution* (or the *nonhomogeneous solution*) of the same equation.

The choice of trial solution depends on the function f(x) on the right-hand side of equation $Ay'' + By' + cy = f(x)$. We will look a three cases:

- Polynomial functions.
- Exponential function.
- Sinusoidal functions.

If $f(x)$ is a ***polynomial*** it is reasonable to guess that there is a particular solution, $y_p(x)$ which is a polynomial in x of the same degree as $f(x)$ (because if y is such a polynomial, then $Ay'' + By' + cy$ is also a polynomial of the same degree.)

If $f(x)$ is an ***exponential*** of the form Ce^{kx}, where C and k are constants, then we use a trial solution of the form $y_p(x) = \mathrm{A}e^{kx}$ and solve for A if possible.

If $f(x)$ is a ***sinusoidal*** of the form C cos (kx) or C sin(kx), where C and k are constants, then we use a trial solution of the form $y_p(x) = \mathrm{A}\cos(kx) + B\sin(kx)$ and solve for A and B if possible.

Troubleshooting

If the trial solution y_p is a solution of the corresponding homogeneous equation, then it cannot be a solution to the non-homogeneous equation. In this case, we multiply the trial solution by($x\ or\ x^2\ or\ x^3$... as necessary) to get a new trial solution that does not satisfy the corresponding homogeneous equation.

Example

Solve the differential equation: $y'' + y' + 2y = x^2$

Solution

We first find the solution of the complementary $y'' + y' + 2y = 0$

Auxiliary equation: $r^2 + r + 2 = 0$

Roots: $(r+1)(r+2) = 0 \longrightarrow r_1 = -1\ , r_2 = -2$

Solution of the complementary

$$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$$

We now need a particular solution $y_p(x)$

We consider a trial solution of the form $\boxed{y_p(x) = \mathrm{A}x^2 + Bx + C}$

Then $\boxed{y'_p(x) = 2Ax + B} \longrightarrow \boxed{y''_p(x) = 2A}$

Substituting $y_p(x)$, $y'_p(x)$ and $y''_p(x)$ in the original equation to get

$$2A + 3(2Ax + B) + 2(\mathrm{A}x^2 + Bx + C) = x^2$$

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = x^2$$

Equating coefficients, we get

$$2A = 1 \longrightarrow A = \frac{1}{2}$$

$$6A + 2B = 0 \longrightarrow 6 \times \frac{1}{2} + 2B = 0 \longrightarrow B = \frac{-3}{2}$$

$$2A + 3B + 2C = 0 \longrightarrow 2 \times \frac{1}{2} + 3 \times \frac{-3}{2} + 2C = 0 \longrightarrow C = \frac{7}{4}$$

Hence a particular solution is given by $y_p(x) = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$

The general solution is given by $y_{G.S} = y_c(x) + y_p(x)$

$$y_{G.S} = C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

Example

Solve the differential equation: $y'' + 9y = e^{-4x}$

Solution

We first find the solution of the complementary $y'' + 9y = 0$

Auxiliary equation: $r^2 + 9 = 0 \longrightarrow r^2 = -9$

Roots: $r_1 = 3i \ , r_2 = -3i$

Solution of the complementary

$$y_c(x) = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x) \longrightarrow y_c(x) = (C_1 \sin(3x) + C_2 \cos(3x))$$

We now need a particular solution $y_p(x)$

We consider a trial solution of the form $\boxed{y_p(x) = Ae^{-4x}}$

Then $\boxed{y'_p(x) = -4Ae^{-4x}}$ and $\boxed{y''_p(x) = 16Ae^{-4x}}$

Substituting $y_p(x)$ and $y''_p(x)$ in the original equation to get

$$16Ae^{-4x} + 9Ae^{-4x} = e^{-4x} \longrightarrow 25Ae^{-4x} = e^{-4x} \longrightarrow A = \frac{1}{25}$$

Hence a particular solution is given by $y_p(x) = \frac{1}{25}e^{-4x}$

The general solution is given by $y_{G.S} = y_c(x) + y_p(x)$

$$y_{G.S} = C_1 \sin(3x) + C_2 \cos(3x) + \frac{1}{25}e^{-4x}$$

Example

Solve the differential equation: $y'' - 4y' - 5y = \cos(2x)$

Solution

We first find the solution of the complementary $y'' - 4y' - 5y = 0$

Auxiliary equation: $r^2 - 4r - 5 = 0$

Roots: $(r-5)(r+1) = 0 \longrightarrow r_1 = 5\ , r_2 = -1$

Solution of the complementary

$$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y_c(x) = C_1 e^{5x} + C_2 e^{-x}$$

We now need a particular solution $y_p(x)$

We consider a trial solution of the form $\boxed{y_p(x) = \mathrm{A}\cos(2\mathrm{x}) + B\sin(2\mathrm{x})}$

Then $\boxed{y'_p(x) = -2A\sin(2x) + 2B\cos(2x)}$

$$\boxed{y''_p(x) = -4A\cos(2x) - 4B\sin(2x)}$$

Substituting $y_p(x)$, $y'_p(x)$ and $y''_p(x)$ in the original equation to get

$$-4A\cos(2x) - 4B\sin(2x) - 4[-2A\sin(2x) + 2B\cos(2x)]$$

$$-5[\mathrm{A}\cos(2\mathrm{x}) + B\sin(2\mathrm{x})\ ] = \cos(2x)$$

$$-4A\cos(2x) - 4B\sin(2x) + 8A\sin(2x) - 8B\cos(2x) - 5A\cos(2x)$$
$$-5B\sin(2x) = \cos(2x)$$

$$-9A - 8B = 1 \dashrightarrow 1 \quad \times 8$$

$$8A - 9B = 0 \dashrightarrow 2 \quad \times 9$$

$$\cancel{-72A} - 64B = 8$$

$$\cancel{72A} - 81B = 0$$

$$-145\,\mathrm{B} = 8 \longrightarrow B = \frac{-8}{145}$$

Substituting the value of B into the first equation, we get $A = \frac{-9}{145}$

Hence a particular solution is given by $y_p(x) = -\frac{9}{145}\cos(2\text{x}) - \frac{8}{145}\sin(2\text{x})$

The general solution is given by $y_{G.S} = y_c(x) + y_p(x)$

$$y_{G.S} = C_1 e^{5x} + C_2 e^{-x} - \frac{9}{145}\cos(2\text{x}) - \frac{8}{145}\sin(2\text{x})$$

Example

Solve the differential equation: $y'' + y' = x - 2$

Solution

We first find the solution of the complementary $y'' + y' = 0$

Auxiliary equation: $r^2 + r = 0$

Roots: $r\,(r+1) = 0 \longrightarrow r_1 = 0\ , r_2 = -1$

Solution of the complementary

$$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y_c(x) = C_1 + C_2 e^{-x}$$

We now need a particular solution $y_p(x)$

We consider a trial solution of the form $y_p(x) = Ax + B$

Is a solution to the corresponding homogeneous equation and therefore cannot be a solution to the non-homogeneous equation.

There is an overlap (the solution B) so we multiply the corresponding trial solution terms by x, to get $\boxed{y_p(x) = \mathrm{A}x^2 + Bx}$

$$\boxed{y'_p(x) = 2Ax + B} \longrightarrow \boxed{y''_p(x) = 2A}$$

Substituting $y'_p(x)$ and $y''_p(x)$ in the original equation to get

$$2A + 2Ax + B = x - 2$$

$$2A = 1 \longrightarrow A = \frac{1}{2}$$

$$2A + B = -2 \longrightarrow 2\left(\frac{1}{2}\right) + B = -2 \longrightarrow B = -3$$

$$y_p(x) = \frac{1}{2}x^2 - 3x$$

Hence a particular solution is given by $y_p(x) = \frac{1}{2}x^2 - 3x$

The general solution is given by $y_{G.S} = y_c(x) + y_p(x)$

$$y_{G.S} = C_1 + C_2e^{-x} + \frac{1}{2}x^2 - 3x$$

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Mathematics

Second Order Differential Equations (Nonhomogeneous) part II

Lecture (8)

Presented by :-
Mahmoud Shakir Wahhab

When g(x) is a sum of several terms

When g(x) is a sum of several functions: $g(x) = g_1(x) + g_2(x) + \cdots + g_n(x)$,

we can break the equation into n parts and solve them separately. Given

$$P(x)y'' + Q(x)y' + R(x)y = g_1(x) + g_2(x) + \cdots + g_n(x)$$

we change it into

$$P(x)y'' + Q(x)y' + R(x)y = g_1(x)$$

$$P(x)y'' + Q(x)y' + R(x)y = g_2(x)$$

$$\vdots$$

$$\vdots$$

$$P(x)y'' + Q(x)y' + R(x)y = g_n(x)$$

Solve them individually to find respective particular solutions $y_1, y_2, \dots\dots, y_n$

Then add up them to get $y = y_1 + y_2 + \cdots + y_n$

Example

Solve the differential equation: $y'' + 4y' + 4y = e^{-2x} + \sin 2x$

Solution

Auxiliary equation: $r^2 + 4r + 4 = 0$

Roots: $(r+2)(r+2) = 0 \longrightarrow r_1 = -2\ , r_2 = -2$

$$y_c(x) = C_1 e^{r_1 x} + C_2 x e^{r_2 x} \longrightarrow \boxed{y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}}$$

$$y_{p1}(x) = \mathrm{A} x^2 e^{-2x}$$

$$y_{p2}(x) = \mathrm{B} \sin 2x + C \cos 2x$$

$$y_p(x) = y_{p1}(x) + y_{p2}(x) \longrightarrow \boxed{y_p(x) = \mathrm{A} x^2 e^{-2x} + \mathrm{B} \sin 2x + C \cos 2x}$$

$$y_p(x) = \mathrm{A}x^2e^{-2x} + \mathrm{B}\sin 2x + C\cos 2x$$

$$\boxed{y'_p(x) = \mathrm{A}(-2x^2e^{-2x} + 2xe^{-2x}) + 2B\cos 2x - 2C\sin 2x}$$

$$y''_p(x) = \mathrm{A}(4x^2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 2e^{-2x}) - 4B\sin 2x - 4C\cos 2x$$

$$\boxed{y''_p(x) = \mathrm{A}(4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x}) - 4B\sin 2x - 4C\cos 2x}$$

$$\because\ y'' + 4y' + 4y = e^{-2x} + \sin 2x$$

$$\mathrm{A}(4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x}) - 4B\sin 2x - 4C\cos 2x +$$

$$4[\mathrm{A}(-2x^2e^{-2x} + 2xe^{-2x}) + 2B\cos 2x - 2C\sin 2x] +$$

$$4(\mathrm{A}x^2e^{-2x} + \mathrm{B}\sin 2x + C\cos 2x) = e^{-2x} + \sin 2x$$

$$4Ax^2e^{-2x} - 8Axe^{-2x} + 2Ae^{-2x} - 4B\sin 2x - 4C\cos 2x - 8Ax^2e^{-2x} + 8Axe^{-2x}$$

$$+8B\cos 2x - 8C\sin 2x + 4Ax^2e^{-2x} + 4B\sin 2x + 4C\cos 2x = e^{-2x} + \sin 2x$$

$$2Ae^{-2x} + 8B\cos 2x - 8C\sin 2x = e^{-2x} + \sin 2x$$

$$2A = 1 \longrightarrow A = \frac{1}{2}$$

$$8B = 0 \longrightarrow B = 0$$

$$-8C = 1 \longrightarrow C = \frac{-1}{8}$$

$$\because y_p(x) = Ax^2e^{-2x} + B\sin 2x + C\cos 2x \longrightarrow y_p(x) = \frac{1}{2}x^2e^{-2x} - \frac{1}{8}\cos 2x$$

$$y_{G.S} = y_c(x) + y_p(x) \longrightarrow \boxed{y_{G.S} = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2}x^2e^{-2x} - \frac{1}{8}\cos 2x}$$

When $g(x)$ is a product of several functions

If $g(x)$ is a product of two or more simple functions, e.g. $g(x) = x^3 e^{5x} cos(4x)$ then our basic choice (before multiplying by x, if necessary) should be a product consist of the corresponding choices of the individual components of g(x). One thing to keep in mind: that there should be only as many undetermined coefficients in Y as there are distinct terms (after expanding the expression and simplifying algebraically).

Example $y'' - 2y' - 3y = x^3 e^{5x} cos(4x)$

$$y_c(x) = C_1 e^{3x} + C_2 e^{-x}$$

$$e^{5x} = Ae^{5x}$$

$$x^3 = \mathrm{B}x^3 + Cx^2 + Dx + E$$

$$\cos(4x) = F\sin(4x) + G\cos(4x)$$

$$e^{5x} x^3 \cos(4x) = Ae^{5x}\{(\mathrm{B}x^3 + Cx^2 + Dx + E\ )(F\sin(4x) + G\cos(4x))\}$$

$$Ae^{5x}\left\{\begin{matrix} \mathrm{B}Fx^3\sin(4x) + \mathrm{BG}x^3\cos(4x) + CFx^2\sin(4x) + CGx^2\cos(4x) + \\ DFx\sin(4x) + DGx\cos(4x) + EF\sin(4x) + EG\cos(4x) \end{matrix}\right\}$$

$$\left\{\begin{matrix} A\mathrm{B}Fe^{5x}x^3\sin(4x) + AB\mathrm{G}e^{5x}x^3\cos(4x) + ACFe^{5x}x^2\sin(4x) + ACGe^{5x}x^2\cos(4x) + \\ ADFe^{5x}x\sin(4x) + ADGe^{5x}x\cos(4x) + AEF\,e^{5x}\sin(4x) + AEGe^{5x}\cos(4x) \end{matrix}\right\}$$

$$\begin{Bmatrix} ABFe^{5x}x^3\sin(4x) + ABGe^{5x}x^3\cos(4x) + ACFe^{5x}x^2\sin(4x) + ACGe^{5x}x^2\cos(4x) + \\ ADFe^{5x}x\sin(4x) + ADGe^{5x}x\cos(4x) + AEF\,e^{5x}\sin(4x) + AEGe^{5x}\cos(4x) \end{Bmatrix}$$

$$y_p(x) = Ae^{5x}x^3\sin(4x) + \mathrm{B}e^{5x}x^3\cos(4x) + Ce^{5x}x^2\sin(4x) + De^{5x}x^2\cos(4x) +$$

$$Ee^{5x}x\sin(4x) + Fe^{5x}x\cos(4x) + G\,e^{5x}\sin(4x) + He^{5x}\cos(4x)$$

Example

Solve the differential equation: $y'' - y' - 6y = e^x \cos x$

Solution

Auxiliary equation: $r^2 - r - 6 = 0$

Roots: $(r-3)(r+2) = 0 \longrightarrow r_1 = 3\ , r_2 = -2$

$$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow \boxed{y_c(x) = C_1 e^{3x} + C_2 e^{-2x}}$$

$$y_p(x) = e^x (A\cos x + B\sin x) \longrightarrow \boxed{y_p(x) = A(e^x \cos x) + B(e^x \sin x)}$$

$$\boxed{y'_p(x) = \mathrm{A}(-e^x \sin x + e^x \cos x) + B(e^x \cos x + e^x \sin x)}$$

$$\boxed{\begin{aligned} y''_p(x) = \mathrm{A}(-e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x) + \\ B(-e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x) \end{aligned}}$$

$$y''_p(x) = \mathrm{A}(-2e^x \sin x) + B(2e^x \cos x)$$

$$y''_p(x) = -2Ae^x \sin x + 2Be^x \cos x$$

$$y'_p(x) = -\mathrm{A}e^x \sin x + Ae^x \cos x + Be^x \cos x + Be^x \sin x$$

$$\because y'' - y' - 6y = e^x \cos x$$

$$-2Ae^x \sin x + 2Be^x \cos x - (-\mathrm{A}e^x \sin x + Ae^x \cos x + Be^x \cos x + Be^x \sin x)$$
$$-6(Ae^x \cos x + Be^x \sin x) = e^x \cos x$$

$$-2Ae^x \sin x + 2Be^x \cos x + \mathrm{A}e^x \sin x - Ae^x \cos x - Be^x \cos x - Be^x \sin x$$
$$-6Ae^x \cos x - 6Be^x \sin x = e^x \cos x$$

$$-Ae^{x}\sin x + Be^{x}\cos x - 7Ae^{x}\cos x - 7Be^{x}\sin x = e^{x}\cos x$$

$$-A - 7B = 0 \dashrightarrow 1$$

$$-7A + B = 1 \dashrightarrow 2 \qquad \times 7$$

$$\begin{aligned} -A - \cancel{7B} &= 0 \\ -49A + \cancel{7B} &= 7 \\ \hline -50A &= 7 \longrightarrow A = \frac{-7}{50} \end{aligned}$$

$$\frac{7}{50} - 7B = 0 \longrightarrow B = \frac{1}{50}$$

$$y_p(x) = Ae^{x}\cos x + Be^{x}\sin x \longrightarrow y_p(x) = \frac{-7}{50}e^{x}\cos x + \frac{1}{50}e^{x}\sin x$$

$$y_{G.S} = y_c(x) + y_p(x) \longrightarrow y_{G.S} = C_1e^{3x} + C_2e^{-2x} - \frac{7}{50}e^{x}\cos x + \frac{1}{50}e^{x}\sin x$$

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Mathematics

Second Order Differential Equations
Variation of Parameters

Lecture (9)

Presented by :-
Mahmoud Shakir Wahhab

Variation of Parameters

A method for solving nonhomogeneous linear differential equations that is more general than the method of undetermined coefficients.

Method of variation of parameters can be used to find a particular solution y_p when

- The coefficients are functions of x.
- The right-hand side function g(x) is any integrable function.

- The complementary function y_c is known. That is, we know the general solution $y_c = C_1\, y_1 + C_2\, y_2$
- To the associated homogeneous ODE, where y_1 and y_2 form the fundamental set of solutions.
- The general solution is $y_{G.S} = y_1 v_1 + y_2 v_2$
- We need two equations to determine v_1 *and* v_2.

$$y_1 v_1' + y_2 v_2' = 0 \longrightarrow 1$$

$$y_1' v_1' + y_2' v_2' = f(x) \longrightarrow 2$$

This is a system of 2 equations in 2 unknowns, v_1' and v_2'

Solve this system for v_1' and v_2'.

We can solve this system using Cramer's rule.

$$y_1 v_1' + y_2 v_2' = 0 \longrightarrow 1$$

$$y_1' v_1' + y_2' v_2' = f(x) \longrightarrow 2$$

$$v_1' = \frac{A_1}{A} \qquad v_2' = \frac{A_2}{A}$$

Integrate v_1' and v_2' to find v_1 and v_2

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$A = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$A_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2 f(x)$$

$$A_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1 f(x)$$

Example

Solve the differential equation: $y'' + y' + 2y = x^2$

Solution

We first find the solution of the complementary $y'' + y' + 2y = 0$

Auxiliary equation: $r^2 + r + 2 = 0$

Roots: $(r+1)(r+2) = 0 \longrightarrow r_1 = -1\ , r_2 = -2$

Solution of the complementary

$$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \longrightarrow y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_{G.S} = y_1 v_1 + y_2 v_2$$

$$y_1 = e^{-x} \longrightarrow y_1' = -e^{-x}$$

$$y_2 = e^{-2x} \longrightarrow y_2' = -2e^{-2x}$$

$$\left.\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= f(x) \end{aligned}\right\} \qquad \begin{aligned} e^{-x} \cdot v_1' + e^{-2x} \cdot v_2' &= 0 \\ -e^{-x} \cdot v_1' - 2e^{-2x} \cdot v_2' &= x^2 \end{aligned}$$

$$A = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} \longrightarrow A = -e^{-3x}$$

$$A_1 = \begin{vmatrix} 0 & e^{-2x} \\ x^2 & -2e^{-2x} \end{vmatrix} = -e^{-2x} \cdot x^2$$

$$e^{-x} \cdot v_1' + e^{-2x} \cdot v_2' = 0$$

$$-e^{-x} \cdot v_1' - 2e^{-2x} \cdot v_2' = x^2$$

$$A_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & x^2 \end{vmatrix} = e^{-x} \cdot x^2$$

$$v_1' = \frac{A_1}{A} \longrightarrow v_1' = \frac{-e^{-2x}x^2}{-e^{-3x}} \longrightarrow v_1' = e^x x^2$$

$$v_1 = \int v_1' \, dx \longrightarrow v_1 = \int e^x x^2 \, dx$$

$$v_1 = x^2 e^x - 2xe^x + 2e^x + K_1$$

$$v_2' = \frac{A_2}{A} \longrightarrow v_2' = \frac{e^{-x} \cdot x^2}{-e^{-3x}} \longrightarrow v_2' = -x^2 e^{2x}$$

$$\begin{array}{ccc} x^2 & & e^x \\ & + & \\ 2x & & e^x \\ & - & \\ 2 & & e^x \\ & + & \\ 0 & & e^x \end{array}$$

$$v_2 = \int v_2'\,dx \longrightarrow v_2 = \int -e^{2x}x^2\,dx$$

$-x^2$	$+$	e^{2x}
$-2x$	$-$	$\frac{1}{2}e^{2x}$
-2	$+$	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

$$v_2 = -\frac{1}{2}x^2e^{2x} + \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x} + K_2$$

$$y_{G.S} = y_1v_1 + y_2v_2$$

$$y_{G.S} = e^{-x}(x^2e^x - 2xe^x + 2e^x + K_1) + e^{-2x}\left(-\frac{1}{2}x^2e^{2x} + \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x} + K_2\right)$$

$$y_{G.S} = x^2 - 2x + 2 + K_1e^{-x} - \frac{1}{2}x^2 + + \frac{1}{2}x - \frac{1}{4} + K_2e^{-2x}$$

$$y_{G.S} = \underbrace{K_1e^{-x} + K_2e^{-2x}}_{y_c(x)} + \underbrace{\frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}}_{y_p(x)}$$

Example

Solve the differential equation: $y'' + y = \tan x$

<u>Solution</u>

$r^2 + 1 = 0 \longrightarrow r^2 = -1 \longrightarrow r = \mp i$

$y_c = e^{\alpha x}(C_1 \cos\beta\, x + C_2 \sin\beta\, x)$

$y_c = e^{0x}(C_1 \cos x + C_2 \sin x) \longrightarrow y_c = (C_1 \cos x + C_2 \sin x)$

$y_1 = \cos x \longrightarrow y_1' = -\sin x$

$y_2 = \sin x \longrightarrow y_2' = \cos x$

$$y_1 = \cos x \longrightarrow y_1' = -\sin x$$

$$y_2 = \sin x \longrightarrow y_2' = \cos x$$

$$\left.\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= f(x) \end{aligned}\right\} \qquad \begin{aligned} \cos x \cdot v_1' + \sin x \cdot v_2' &= 0 \\ -\sin x \cdot v_1' + \cos x \cdot v_2' &= \tan x \end{aligned}$$

$$A = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \longrightarrow A = 1$$

$$A_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \cdot \tan x$$

$$A_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \cdot \tan x \longrightarrow A_2 = \sin x$$

$$v_1' = \frac{A_1}{A} \longrightarrow v_1' = -\sin x \cdot \tan x$$

$$v_2' = \frac{A_2}{A} \longrightarrow v_2' = \sin x$$

$$v_1 = \int v_1'\, dx \longrightarrow v_1 = -\int \sin x \tan x\, dx$$

$$v_1 = -\int \sin x \cdot \frac{\sin x}{\cos x} dx \longrightarrow -\int \frac{(1 - \cos^2 x)}{\cos x} dx$$

$$-\left[\int \sec x\, dx - \int \cos x\, dx\right] \longrightarrow -[\ln|\sec x + \tan x| - \sin x] + k_1$$

$$v_1 = \sin x - \ln|\sec x + \tan x| + k_1$$

$$v_2 = \int v_2' \, dx \longrightarrow v_2 = \int \sin x \, dx \longrightarrow v_2 = -\cos x + K_2$$

$$y_{G.S} = y_1 v_1 + y_2 v_2$$

$$y_{G.S} = \cos x \left(\sin x - \ln|\sec x + \tan x| + k_1\right) + \sin x \left(-\cos x + K_2\right)$$

$$y_{G.S} = \cancel{\sin x \cos x} - \cos x \ln|\sec x + \tan x| + k_1 \cos x - \cancel{\sin x \cos x} + K_2 \sin x$$

$$y_{G.S} = \underbrace{k_1 \cos x + K_2 \sin x}_{y_c(x)} \underbrace{- \cos x \ln|\sec x + \tan x|}_{y_p(x)}$$

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