وزارة التعليم العالي والبحث العلمي في العراق

الجامعة التقنية الشمالية

الكلية التقنية/كركوك

قسم تقنيات هندسة ميكانيك القدرة

المرحلة الاولى

مادة الرياضيـــات I

إعداد مدرس المادة

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Ministry of Higher Education and Scientific Research / Iraq

Northern Technical University

Technical College / Kirkuk

Mechanical Power Techniques Engineering Dept

First Year

MATHEMATICS I

Prepared by subject lecturer

Assist. Lec. Intisar K. Saleh

Subject	Target students	Hours in a week			Units
		Theory	Practical	Total	
MATHEMATICS I	First year students	3	-	3	6

The Aims

1. **General aims:**
   This course will provide the student with principles of first part of mathematics (CALCULUS) like matrices, trigonometry, conics, vectors, limits, derivatives and methods of integration, with their engineering applications.

2. **Special aims:** **The students can be able to;**

A – Provides the student with a comprehensive, thorough, and up-to-date treatment of engineering mathematics,
B – Solving the mathematical equations to get the unknown variables, using matrices,
C – Gives an idea about limits and there engineering applications,
D – Provides the student with introduction to matrices and their calculations with the methods of solving simultaneous equation,
E – Provides the student with introduction to derivatives and methods of integrations.

توزيع الدرجات خلال السنة:

- الفصل الأول: ويتضمن 20 درجة امتحان فصلي نظري. (نصف السنة).
- الفصل الثاني: ويتضمن 20 درجة امتحان فصلي نظري.
- أعمال السنة: 10 درجات (امتحانات سريعة + حضور الطالب)

(فيكون السعي السنوي من 50 درجة)

- الامتحان النهائي: 50 درجة

(فتكون الدرجة النهائية من 100 درجة)

Syllabus:

المنهاج:

هيئة التعليم التقني الكلية التقنية – بغداد القسم : هندسةالتبريد والتكييف	النظام السنوي 30أسبوع	عدد الساعات الاسبوعية	ن 3	ع –	م 3	عددالوحدات 6
المـــرحلـــة الاولـــى	مفـــردات مـــادة الـــريـــاضيـــات –1	الجزء النظري				

الهـــدف مـــن المـــادة

تعريف الطالب على المبادئ الأساسية والمتقدمة في التفاضل والتكامل وتطبيقاتها المختلفة لتنمية وتطوير قدراته الذهنية عند حل التمارين وربط المعطيات مع معلوماته للوصول الى حل المسألة والاستفادة منها في المواد العلمية الأخرى.

الأسبـــوع	مفـــردات المـــادة
2–1	المحددات وخواصها – محددات من الدرجة n .حل المعادلات الخطية بطريقة كرامر – تطبيقات على المحددات
4–3	الدوال المثلثية – العلاقات المثلثية ورسم منحنيات الدوال – التطبيقات والمعادلات المثلثية – تطبيقات متنوعة على الدوال المثلثية .
6–5	المتجهات – العمليات الحسابية للمتجهات في الفضاء الثنائي والثلاثي – وحدة المتجهات المتعامدة – مقياس المتجه – الضرب القياسي والأتجاهي والمساقط – أيجاد مساحة الأشكال بطريقة المتجهات – تطبيقات ميكانيكية على المتجهات
8–7	الدالة والغاية – الغايات – غاية الدوال الجبرية والمثلثية وغاية الدوال عندما تقترب من (∞) - تطبيقات على الغايات.
9	نظرية المشتقة – الدوال المركبة – مشتقات الدوال الجبرية والمثلثية والدوال الضمنية.
10	الدوال القياسية – قاعدة السلسلة – تطبيقات ميكانيكية على المشتقة .
11	الدالة العكسية – مشتقة الدوال العكسية المثلثية – تطبيقات متنوعة .
14–12	مشتقة الدوال اللوغارتمية والأسية- دوال القطع الزائد – مشتقة الدوال الزائدية ومعكوس الدالة- العلاقات والرسم والمعكوس للدوال الزائدية – تطبيقات فيزياوية وميكانيكية .
18–15	التكامل – نظرية التكامل – التكامل المحدد والتكامل غير المحدد- تكامل الدوال المثلثية والعكسية – تكامل الدوال الأسية اللوغاريتمية – تكامل دوال القطع الزائد والعكسية – التكامل المعتل وقاعدة لوبيتال .
19	طرق التكامل :– التكامل بالتجزئة .
20	التكامل بطريقة تجزئة الكسور.
21	التكامل بطريقة تعويض الدالة المثلثية .
22	التكامل بطريقة اكمال المربع والفرضية ...... الخ .
23	تطبيقات التكامل الفيزياوية والهندسية .
24	المساحة تحت منحي وبين منحنيين.
27–25	الحجوم الدورانية – طول قوس منحني.
28	المعادلات التفاضلية المبسطة .
30–29	المساحة التقريبية بأستخدام قاعدة شبه المنحرف وسمبسون – الطريقة العددية في التكامل – تطبيقات .

References:

المصادر:

1- "CALCULUS", by George. B. Thomas.

2- "Engineering Mathematics", by John Bird.

3- Any other Mathematics book.

Addition with Infinity

Infinity Plus a Number $\infty \pm k = \infty$ (k is any number)

Infinity Plus Infinity $\infty + \infty = \infty$

Infinity Minus Infinity $\infty - \infty \rightarrow$ Indeterminate Form

Multiplication with Infinity

Infinity by a Number $\infty \cdot (\pm k) = \pm\infty \quad if\ k \neq 0$

Infinity by Infinity $\infty \cdot \infty = \infty$

Infinity by Zero $0 \cdot \infty \rightarrow Ind$

Division with Infinity and Zero

Zero over a Number $\frac{0}{k} = 0$

A Number over Zero $\frac{k}{0} = \pm\infty$

A Number over Infinity $\frac{k}{\infty} = 0$

Infinity over a Number $\frac{\infty}{k} = \infty$

Zero over Infinity $\frac{0}{\infty} = 0$

Infinity over Zero $\frac{\infty}{0} = \infty$

Zero over Zero $\frac{0}{0} \rightarrow Ind$

Infinity over Infinity $\frac{\infty}{\infty} \rightarrow Ind$

Powers with Infinity and Zero

A Number to the Zero Power $k^0 = 1$

Zero to the Power Zero $0^0 \rightarrow Ind$

Infinity to the Power Zero $\infty^0 \rightarrow Ind$

Zero to the Power of a Number $0^k = \begin{cases} 0 & \text{if } k>0 \\ \infty & \text{if } k<0 \end{cases}$

A Number to the Power of Infinity $k^\infty = \begin{cases} \infty & \text{si } K>1 \\ 0 & \text{si } 0<k<1 \end{cases}$

Zero to the Power of Infinity $0^\infty = 0$

Infinity to the Power of Infinity $\infty^\infty = \infty$

One to the Power of Infinity $1^\infty \rightarrow Ind$

MATRICES

Addition and Scalar Multiplication for Matrices:

A Matrix: Is a rectangular array of numbers or functions which enclosed in brackets.

For example:

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

are matrices. The numbers (or functions) inside the matrix are called **entries** or, less commonly, **elements** of matrix. The first matrix in up has two **rows**, which are the horizontal lines of entries. Furthermore, it has three **columns**, which are the vertical lines of entries. The second and third matrices are **square matrices**, which mean that each has as many rows as columns 3 and 2, respectively. The entries of the second matrix have two indices, signifying their location within the matrix. The first index is the number of the row and the second is the number of the column, so that together the entry's position is uniquely identified. For example, (read *a two three*) is in Row 2 and Column 3, etc.

Matrices having just a single row or column are called **vectors**. Thus, the fourth matrix has just one row and is called a **row vector**. The last matrix has just one column and is called a **column vector**.

Now, if we are given a system of linear equations, briefly a linear system, such as:

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + x_3 &= 10 \end{aligned}$$

where x_1, x_2, and x_3 are **unknowns**. We form the **coefficient matrix**, call it **A**, by listing the coefficients of the unknowns in the position in which they appear in the linear equations.

$$\begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 10 \end{bmatrix}$$

OR: $\mathbf{A}\ \boldsymbol{x} = \mathbf{b}$

Note: The symbol used for denoting a matrix such as A is either **A** or $\bar{A}$

General Notation af a matrix:

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix **A** has **m** rows and **n** columns which are called **size** of the matrix.

Now, for the matrices in Example#1, the sizes are 2*3, 3*3, 2*2, 1*3, and 2*1.

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

If **m=n**, we call **A** as **n*n square matrix**.

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector.

Thus, (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Example#1:

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

The following matrices are all different. Explain!

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$$

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

Example#2:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Example#3:

If $\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$, $\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$, $0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rules for Matrix Addition and Scalar Multiplication. From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero. If $m = 1$ or $n = 1$, this is a vector, called a **zero vector**.

Also,

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

Matrix Multiplication:

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{matrix} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{matrix}$$

The condition $r = n$ means that the second factor, $\mathbf{B}$, must have as many rows as the first factor has columns, namely n. A diagram of sizes that shows when matrix multiplication is possible is as follows:

$$\begin{matrix} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{matrix}$$

$$m = 4 \left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2} \right\} m = 4$$

Notations in a product **AB** = **C**

Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined.

نظرب الصف الاول للمصفوفة الاولى في جميع اعمدة المصفوفة الثانية وبهذا نحصل على الصف الاول لمصفوفة النتيجة..وبعدها ننزل للصف الثاني و هكذا

Example#1:

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix} \quad \text{whereas} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \quad \text{is undefined.}$$

Example#2:

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

CAUTION! Matrix Multiplication Is Not Commutative, AB ≠ BA in General

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

So,

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$ *written* $k\mathbf{AB}$ *or* $\mathbf{A}k\mathbf{B}$

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ *written* $\mathbf{ABC}$

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

Determinant of a Matrix (or the value of a matrix):

Determinants play an important role in finding the inverse of a matrix and also in solving systems of linear equations. In the following we assume that we have a square matrix (rows = columns) or (m = n). The determinant of a matrix A will be denoted by det(A) or |A|. Firstly the determinant of a 2×2 and 3×3 matrix will be introduced, then the n×n case will be shown.

1) Determinant of 2×2 matrix:

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example#1: Find the determinant of the following matrix; $A = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$

Solution: $\det(A) = \begin{vmatrix} 3 & 8 \\ 4 & 6 \end{vmatrix} = 3 * 6 - 8 * 4 = 18 - 32 = \mathbf{-14}$

2) Determinant of 3×3 matrix:

$$\det \begin{bmatrix} \overset{+}{a_1} & \overset{-}{a_2} & \overset{+}{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) \quad - a_2 (b_1 c_3 - b_3 c_1) \quad + a_3 (b_1 c_2 - b_2 c_1)$$

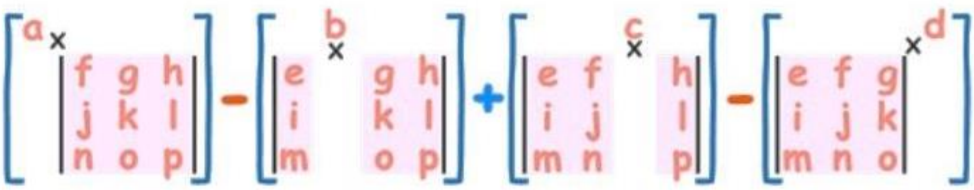
Example#1: Find the determinant of the following matrix; $A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$

Solution: $\det(A) = \begin{vmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{vmatrix} = 6 * (-2 * 7 - 5 * 8) - 1 * (4 * 7 - 5 * 2) + 1 * (4 * 8 — 2 * 2) = 6 * (-54) - 1 * (18) + 1 * (36) = \mathbf{-306}$

3) Determinant of 4×4 matrix:

The pattern continues for 4×4 matrices:

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the + - + - pattern (+a... -b... +c... -d...). This is important to remember.

Note: We can extend these rules to get the determinant of any n x n matrix.

SOLUTION OF SIMULTANEOUS EQUATIONS USING CRAMER'S RULE

There are many forms of Cramer's Rule. One of them is the following:

Cramers rule states that if

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

then $\boldsymbol{x} = \dfrac{\boldsymbol{D_x}}{\boldsymbol{D}}$, $\boldsymbol{y} = \dfrac{\boldsymbol{D_y}}{\boldsymbol{D}}$ and $\boldsymbol{z} = \dfrac{\boldsymbol{D_z}}{\boldsymbol{D}}$

where $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

i.e. the x-column has been replaced by the R.H.S. b column,

$$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

i.e. the y-column has been replaced by the R.H.S. b column,

$$D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

i.e. the z-column has been replaced by the R.H.S. b column.

Example#1: Solve the following simultaneous equations using Cramer's rule;

$$x + y + z = 4$$

$$2x - 3y + 4z = 33$$

$$3x - 2y - 2z = 2$$

Solution:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{vmatrix}$$

$$= 1(6 - (-8)) - 1((-4) - 12)$$
$$+ 1((-4) - (-9)) = 14 + 16 + 5 = \mathbf{35}$$

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 33 & -3 & 4 \\ 2 & -2 & -2 \end{vmatrix}$$

$$= 4(6 - (-8)) - 1((-66) - 8)$$
$$+ 1((-66) - (-6)) = 56 + 74 - 60 = \mathbf{70}$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 33 & 4 \\ 3 & 2 & -2 \end{vmatrix}$$

$$= 1((-66) - 8) - 4((-4) - 12) + 1(4 - 99)$$
$$= -74 + 64 - 95 = \mathbf{-105}$$

$$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -3 & 33 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= 1((-6) - (-66)) - 1(4 - 99)$$
$$+ 4((-4) - (-9)) = 60 + 95 + 20 = \mathbf{175}$$

Hence

$$x = \frac{D_x}{D} = \frac{70}{35} = \mathbf{2},\ y = \frac{D_y}{D} = \frac{-105}{35} = \mathbf{-3}$$

and $z = \dfrac{D_z}{D} = \dfrac{175}{35} = \mathbf{5}$

H.W.: Using Cramer's rule, calculate the unknown variables (x, y, and z) for the following system of linear equations:

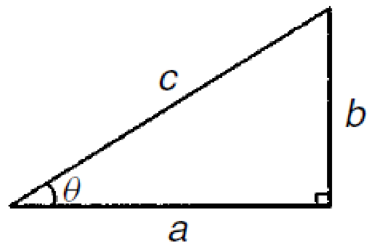
$$2x + y + z = 3$$
$$x - y - z = 0$$
$$x + 2y + z = 0$$

Answer: x = 1, y = -2, z = 3

TRIGONOMETRY

Trigonometry: is the branch of mathematics that deals with the measurement of sides and angles of triangles, and their relationship with each other.

The theorem of Pythagoras: *"In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides".*



$$c^2 = a^2 + b^2$$

Knowing that;

$$\text{sine}\,\theta = \frac{\text{opposite side}}{\text{hypotenuse}}, \quad \text{i.e. } \sin\theta = \frac{b}{c}$$

$$\text{cosine}\,\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}, \quad \text{i.e. } \cos\theta = \frac{a}{c}$$

$$\text{tangent}\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}, \quad \text{i.e. } \tan\theta = \frac{b}{a}$$

$$\text{secant}\,\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}, \quad \text{i.e. } \sec\theta = \frac{c}{a}$$

$$\text{cosecant}\,\theta = \frac{\text{hypotenuse}}{\text{opposite side}}, \quad \text{i.e. } \text{cosec}\,\theta = \frac{c}{b}$$

$$\text{cotangent}\,\theta = \frac{\text{adjacent side}}{\text{opposite side}}, \quad \text{i.e. } \cot\theta = \frac{a}{b}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \tan\theta,$$

$$\text{i.e. } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \cot\theta,$$

$$\text{i.e. } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

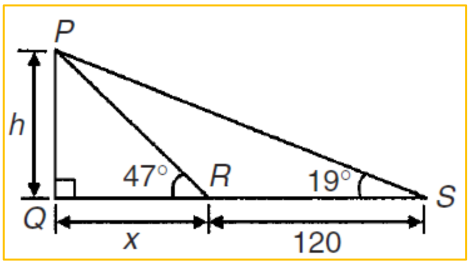
$$\text{cosec}\,\theta = \frac{1}{\sin\theta}$$ (Note 's' and 'c' go together)

$$\cot\theta = \frac{1}{\tan\theta}$$

Also, $\sin\theta = \cos(90^\circ - \theta)$ and $\cos\theta = \sin(90^\circ - \theta)$

Secants, cosecants and cotangents are called the **reciprocal ratios.**

Example#1: A surveyor at position (S) measured the angle of elevation of the top (P) of a perpendicular building, which was 19°. He moved 120 m nearer the building at position (R) and found that the angle of elevation is now 47°. Determine the height of the building (h).



Solution: In triangle PQS, $\tan 19^\circ = \dfrac{h}{x+120}$

hence $h = \tan 19^\circ (x+120)$,

i.e. $h = 0.3443(x+120)$ (1)

In triangle PQR, $\tan 47^\circ = \dfrac{h}{x}$

hence $h = \tan 47^\circ (x)$, i.e. $h = 1.0724x$ (2)

Equating equations (1) and (2) gives:

$$0.3443(x+120) = 1.0724x$$

$$0.3443x + (0.3443)(120) = 1.0724x$$

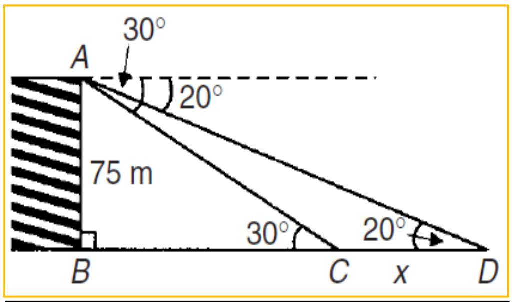
$$(0.3443)(120) = (1.0724 - 0.3443)x$$

$$41.316 = 0.7281x$$

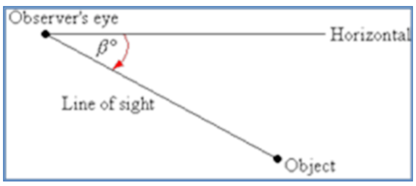
$$x = \frac{41.316}{0.7281} = 56.74 \text{ m}$$

From equation (2), **height of building, h** $= 1.0724x$ $= 1.0724(56.74) =$ **60.85 m**

Example#2: The angle of depression of a ship viewed at a particular instant at position (C) from the top of a 75 m vertical cliff (A) is 30°. Find the horizontal distance of the ship from the base of the cliff (B) at this instant. The ship is sailing away from the cliff at constant speed and 1 minute later its angle of depression (at D) from the top of the cliff is 20°. Determine the speed of the ship in km/h.



Angle of Depression (Angles of Elevation): Is the angle of elevation of an object as seen by an observer or the angle between the horizontal and the line from the object to the observer's eye (the line of sight).



Solution: $\tan 30^\circ = \dfrac{AB}{BC} = \dfrac{75}{BC}$

hence $BC = \dfrac{75}{\tan 30^\circ} = \dfrac{75}{0.5774}$

$=$ **129.9 m**

$=$ **initial position of ship from base of cliff**

In triangle ABD,

$$\tan 20^\circ = \frac{AB}{BD} = \frac{75}{BC + CD} = \frac{75}{129.9 + x}$$

Hence

$$129.9 + x = \frac{75}{\tan 20^\circ} = \frac{75}{0.3640} = 206.0 \text{ m}$$

from which,

$$x = 206.0 - 129.9 = 76.1 \text{ m}$$

Thus the ship sails 76.1 m in 1 minute, i.e. 60 s, hence,

$$\textbf{speed of ship} = \frac{\text{distance}}{\text{time}} = \frac{76.1}{60} \text{ m/s}$$

$$= \frac{76.1 \times 60 \times 60}{60 \times 1000} \text{ km/h}$$

$$= \mathbf{4.57 \text{ km/h}}$$

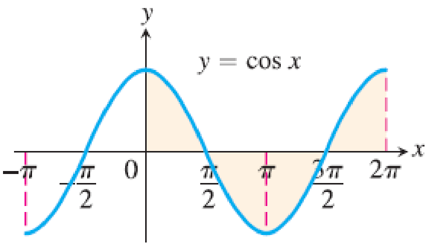
H.W.#1: From a point on horizontal ground a surveyor measures the angle of elevation of the top of a flagpole as 18° 40'. He moves 50 m nearer to the flagpole and measures the angle of elevation as 26° 22'. Determine the height of the flagpole. **Ans. [53.0 m]**

H.W.#2: From a window 4.2 m above horizontal ground the angle of depression of the foot of a building across the road is 24° and the angle of elevation of the top of the building is 34°. Determine, and correct to the nearest centimetre, the width of the road and the height of the building. **Ans. [width = 9.43 m, height = 10.56 m]**

Periodicity and Graphs of the Trigonometric Functions:

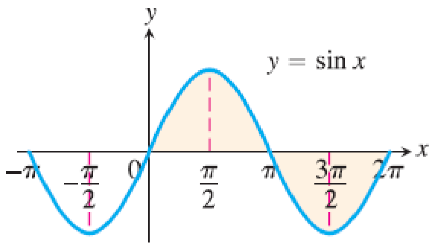
When an angle of measure Θ and an angle of measure $\Theta + 2\pi$ are in standard position, the two angles have the same trigonometric function values: $\sin(\Theta+2\pi)=\sin\Theta$, $\cos(\Theta+2\pi)=\cos\Theta$, $\tan(\Theta+2\pi)=\tan\Theta$, and so on. Similarly, $\sin(\Theta-2\pi)=\sin\Theta$, $\cos(\Theta-2\pi)=\cos\Theta$, $\tan(\Theta-2\pi)=\tan\Theta$, and so on. We describe this repeating behaviour for the six basic trigonometric functions as "**Periodic**"

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.



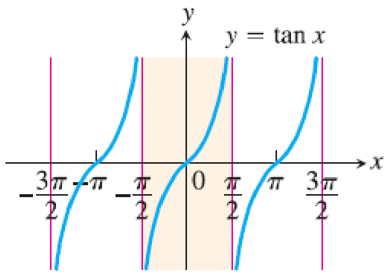
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



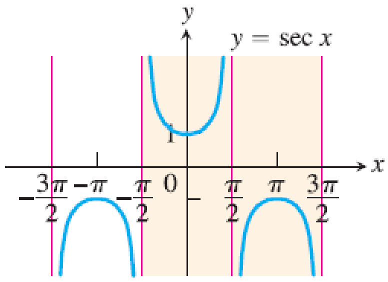
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



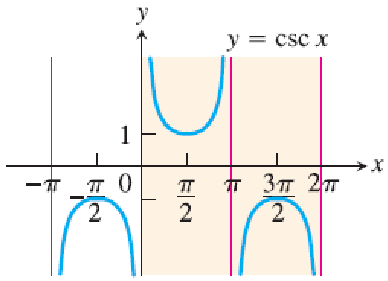
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



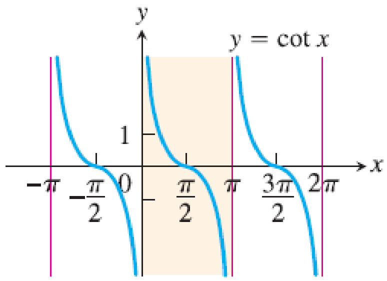
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

Periods of Trigonometric Functions

Period π : $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π : $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

SOME TRIGONOMETRIC IDENTITIES:

$$\sin^2\Theta + \cos^2\Theta = 1$$
$$\tan^2\Theta + 1 = \sec^2\Theta$$
$$1 + \cot^2\Theta = \csc^2\Theta$$
$$\sin(-\Theta) = -\sin\Theta$$
$$\cos(-\Theta) = \cos\Theta$$
$$\tan(-\Theta) = -\tan\Theta$$
$$\sin(\Theta+2\pi) = \sin\Theta$$
$$\cos(\Theta+2\pi) = \cos\Theta$$
$$\tan(\Theta+2\pi) = \tan\Theta$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

cos(A+B) = cosA cosB + sinA sinB

cos(A-B) = cosA cosB – sinA sinB

tan(A+B) = (tanA+tanB)/(1-tanA tanB)

tan(A-B) = (tanA – tanB)/(1+tanA tanB)

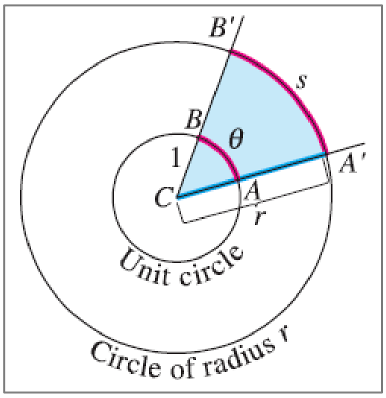
sin2Θ = 2sinΘ cosΘ

cos2Θ = $\cos^2$Θ-$\sin^2$Θ = 2$\cos^2$Θ-1 = 1-2$\sin^2$Θ

$\sin^2$ (Θ /2) = (1-cosΘ)/2

$\cos^2$ (Θ /2) = (1+cosΘ)/2

Angels:



Angle θ is measured in degrees or radians.

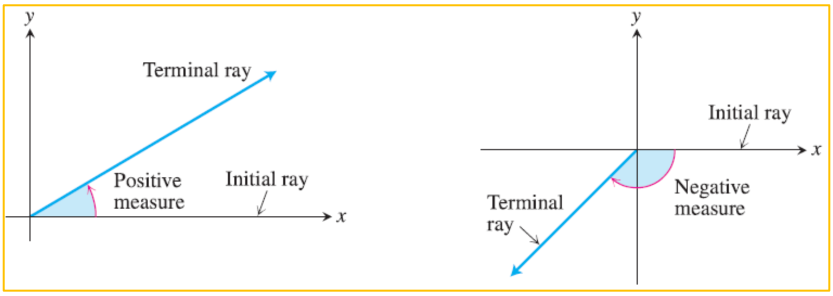
$s = r\,\theta$ (θ here is in radians)

$$\pi \text{ radians} = 180°$$

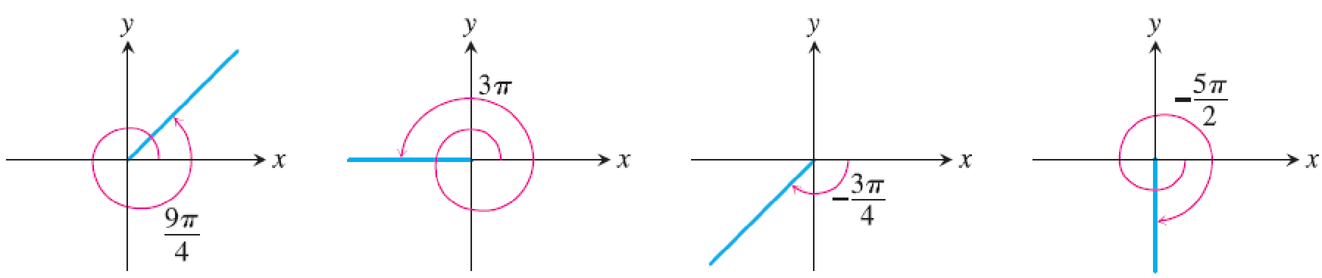
and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

Positive and negative angles:



For example:



H. W.:

Prove the following trigonometric identities:

1. $\sin x \cot x = \cos x$
2. $\dfrac{1}{\sqrt{1-\cos^2\theta}} = \operatorname{cosec}\theta$
3. $2\cos^2 A - 1 = \cos^2 A - \sin^2 A$
4. $\dfrac{\cos x - \cos^3 x}{\sin x} = \sin x \cos x$
5. $(1+\cot\theta)^2 + (1-\cot\theta)^2 = 2\operatorname{cosec}^2\theta$
6. $\dfrac{\sin^2 x(\sec x + \operatorname{cosec} x)}{\cos x \tan x} = 1 + \tan x$

Vector Analysis

A **scalar** is a quantity that is determined by its magnitude. It takes only a numerical value, i.e., a number. Examples of scalars are time, temperature, length, distance, speed, density, energy, mass, and voltage.

A **vector** is a quantity that has both magnitude and direction. We can say that a vector is an ***arrow*** or a ***directed line segment***. For example, a velocity vector has length or magnitude, which is speed, and direction, which indicates the direction of motion. Typical examples of vectors are displacement, velocity, and force.

We refer to vectors by either bold letter like (**A, AB,** or **a**) or by a line like ($\overline{A}$, $\overline{AB}$, or $\overline{a}$) or by an arrow like ($\vec{A}$, $\overrightarrow{AB}$, or $\vec{a}$).

Equality of Vectors: Two vectors **a** and **b** are equal, written **a** = **b**, if they have the same length and the same direction.

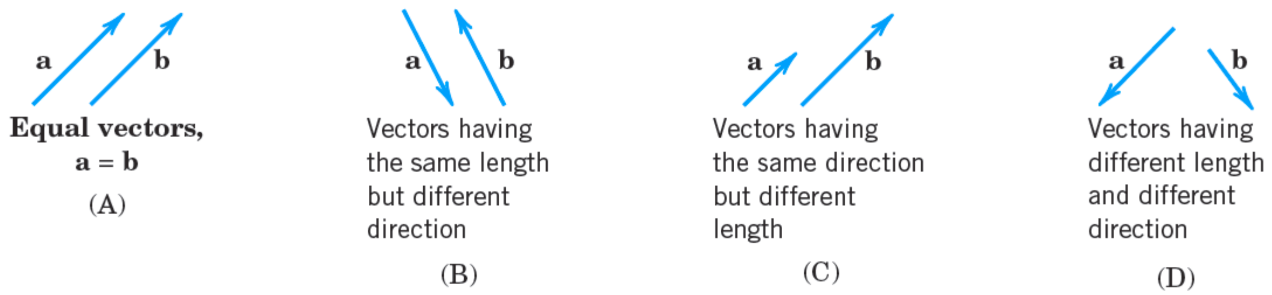
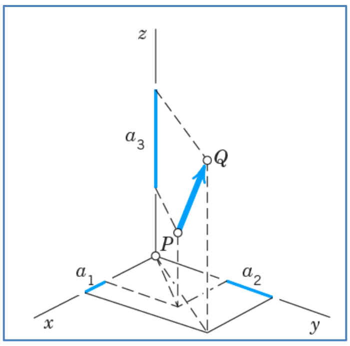


Figure (A) shows Equal Vectors, and Figures (B-C-D) Show Different Vectors

Components of a Vector: Let the vector **PQ** shown in figure, then a_1, a_2, and a_3 are called "Components of the Vector in Cartesian Coordinates", and are calculated as:



$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1$$

Using Pythagorean Theorem, the "**Length**" of the vector **a** (**PQ**) is:

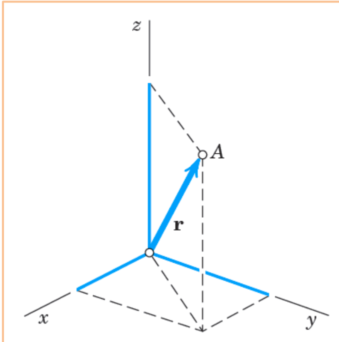
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example#1: Calculate the components and length of 3D vector **PQ** with initial point P(4,0,2) and terminal (end) point Q(6,-1,2).

Solution: $a_1 = x_2 - x_1 = 6 - 4 = 2$, $a_2 = y_2 - y_1 = -1 - 0 = -1$, $a_3 = z_2 - zx_1 = 2 - 2 = 0$

….. then the length is: $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$

***Position Vector*:** Is the vector with origin (0,0,0). Thus the components of **r** will be *x,y,z* which are the coordinates of the terminal point A, as shown in figure.

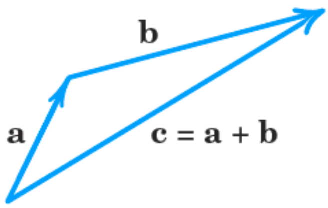


Vectors Addition

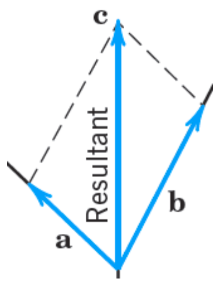
Either Mathematically; the sum of two vectors **a** = [a_1,a_2,a_3] and **b** = [b_1,b_2,b_3] is obtained by getting a new vector by adding the corresponding components;

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3].$$

OR Graphically; there are two methods: **Tip-to-Tail** Method and **Parallelogram** Method



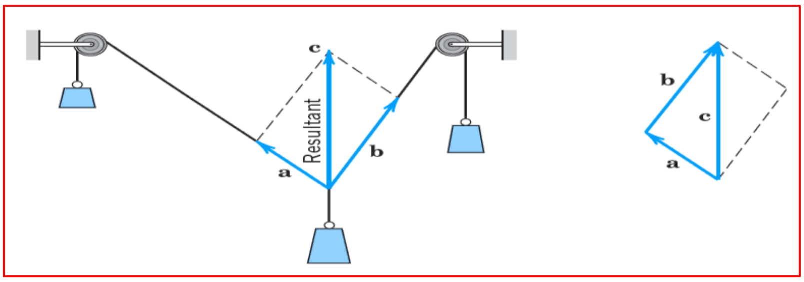
Tip-to-Tail Method



Parallelogram Method

Mechanics Example:

(Resultant c of two forces a&b)



Basic Properties of Vector Addition:

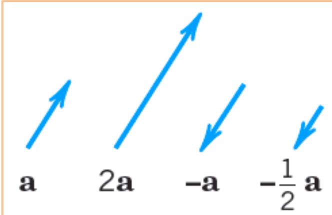
Let (**a, b, u, v,** and **w** are vectors)

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$
$$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$$
$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

Scalar Multiplication (by a number)

The product $c\mathbf{a}$ of a vector $\mathbf{a} = [a_1, a_2, a_3]$ and a scalar c (real number) is:

$$c\mathbf{a} = [ca_1, ca_2, ca_3].$$



So, we multiply c by each component.

Basic Properties of Scalar Multiplication:

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$$
$$(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$$
$$c(k\mathbf{a}) = (ck)\mathbf{a}$$
$$1\mathbf{a} = \mathbf{a}.$$

Example#2: Let two 3D vectors **a** = [4,0,1] and **b** = [2,-5,1/3]. Find –**a**, 7**a**, **a**+**b**, and 2(**a**-**b**).

Solution: $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \frac{4}{3}]$, and

$$2(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{3}] = [4, 10, \tfrac{4}{3}] = 2\mathbf{a} - 2\mathbf{b}.$$

Unit Vector: A vector **a** of length 1 is called a unit vector. The standard unit vectors are **i** = (1,0,0), **j** = (0,1,0), and **k** = (0,0,1).

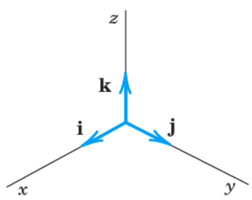
Any vector **a** = (a_1, a_2, a_3) can be written as a linear combination of the standard unit vectors as follows:

a = $(a_1,a_2,a_3) = (a_1,0,0) + (0,a_2,0) + (0,0,a_3)$

$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

Therefore,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$



Therefore, using ***i, j, k*** notations, the two vectors (**a** & **b**) in Example#2 will be:

$$\mathbf{a} = 4\mathbf{i} + \mathbf{k},\ \mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + \tfrac{1}{3}\mathbf{k}, \text{ and so on.}$$

H.W.: Let $\mathbf{a} = [3, 2, 0] = 3\mathbf{i} + 2\mathbf{j}$; $\mathbf{b} = [-4, 6, 0] = 4\mathbf{i} + 6\mathbf{j}$, $\mathbf{c} = [5, -1, 8] = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$, $\mathbf{d} = [0, 0, 4] = 4\mathbf{k}$.

Find:

$2\mathbf{a}$, $\frac{1}{2}\mathbf{a}$, $-\mathbf{a}$

$(\mathbf{a} + \mathbf{b}) + \mathbf{c}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c})$

$\mathbf{b} + \mathbf{c}$, $\mathbf{c} + \mathbf{b}$

$3\mathbf{c} - 6\mathbf{d}$, $3(\mathbf{c} - 2\mathbf{d})$

$7(\mathbf{c} - \mathbf{b})$, $7\mathbf{c} - 7\mathbf{b}$

$\frac{9}{2}\mathbf{a} - 3\mathbf{c}$, $9(\frac{1}{2}\mathbf{a} - \frac{1}{3}\mathbf{c})$

$(7 - 3)\mathbf{a}$, $7\mathbf{a} - 3\mathbf{a}$

$4\mathbf{a} + 3\mathbf{b}$, $-4\mathbf{a} - 3\mathbf{b}$

Dot Product (Inner Product) of Two Vectors

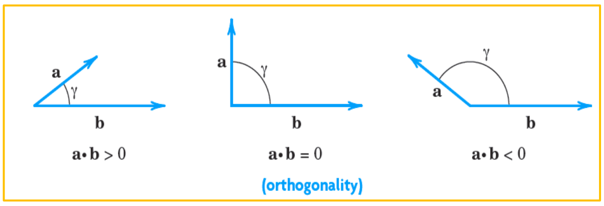
The dot (inner) product of two vectors **a** & **b** is the product of their lengths times cosine of the angle between them, and it is a **scalar** quantity. Thus;

$$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\gamma \quad \text{if} \quad \mathbf{a}\neq\mathbf{0}, \mathbf{b}\neq\mathbf{0}$$

$$\mathbf{a}\bullet\mathbf{b} = 0 \quad \text{if} \quad \mathbf{a}=\mathbf{0} \text{ or } \mathbf{b}=\mathbf{0} \text{ or } \gamma = 90^o$$

Knowing that the "**length**" of vector **a** is:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$



(cosine of γ may be +ve, 0, or –ve)

THEOREM1:

Orthogonality Criterion

The inner product of two nonzero vectors is 0 *if and only if these vectors are perpendicular.*

Therefore, the angle γ between any two nonzero vectors, is:

$$\cos\gamma = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a}\bullet\mathbf{b}}{\sqrt{\mathbf{a}\bullet\mathbf{a}}\sqrt{\mathbf{b}\bullet\mathbf{b}}}.$$

Basic Properties of Dot Product:

For vectors ***a, b,*** and ***c*** and scalars q_1, and q_2:

$$(q_1\mathbf{a} + q_2\mathbf{b})\bullet\mathbf{c} = q_1\mathbf{a}\bullet\mathbf{c} + q_1\mathbf{b}\bullet\mathbf{c}$$

$$\mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{a}$$

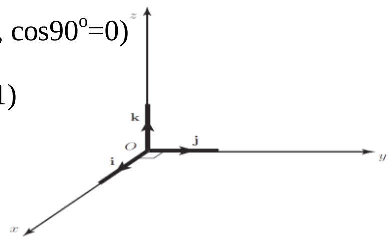
$$\mathbf{a}\bullet\mathbf{a} \geqq 0$$

$$\mathbf{a}\bullet\mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0}$$

Also, If **i, j** and **k** are unit vectors in the directions of the **x, y** and **z** axes, respectively, then:

i . j = 0 **i . k** = 0 **j . k** = 0 (because they are perpendicular, γ = 90°, cos90°=0)

i . i = 1 **j . j** = 1 **k . k** = 1 (because they are parallel, γ = 0, cos0°=1)



Suppose (**a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**) then:

a . b = (a_1**i** + a_2**j** + a_3**k**) . (b_1**i** + b_2**j** + b_3**k**)

$= a_1i \cdot (b_1i + b_2j + b_3k) + a_2j \cdot (b_1i + b_2j + b_3k) + a_3k \cdot (b_1i + b_2j + b_3k)$

$= a_1i \cdot b_1i + a_1i \cdot b_2j + a_1i \cdot b_3k + a_2j \cdot b_1i + a_2j \cdot b_2j + a_2j \cdot b_3k + a_3k \cdot b_1i + a_3k \cdot b_2j + a_3k \cdot b_3k$

$= a_1b_1i \cdot i + a_1b_2i \cdot j + a_1b_3i \cdot k + a_2b_1j \cdot i + a_2b_2j \cdot j + a_2b_3j \cdot k + a_3b_1k \cdot i + a_3b_2k \cdot j + a_3b_3k \cdot k$

Therefore; the dot product of two vectors (**a** & **b**) means:

$$\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Example#1: Find the inner product and the lengths of 3D (R^3) vectors **a** = [1,2,0] and **b** = [3,-2,1], then find the angle γ between these two vectors.

Solution: $\mathbf{a} \bullet \mathbf{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$, $|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{5}$, $|\mathbf{b}| = \sqrt{\mathbf{b} \bullet \mathbf{b}} = \sqrt{14}$, and

$$\gamma = \arccos \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \arccos(-0.11952) = 1.69061 = 96.865°.$$

Example#2: Find the dot (scalar) product of 3D vectors **a = 4i+3j+7k** and **b = 2i+5j+4k**

Solution: **a . b** = (4)*(2) + (3)*(5) + (7)*(4)

= 8 + 15 + 28

= **51**

Example#3: Find the dot (scalar) product of 3D vectors **a = -6i+3j-11k** and **b = 12i+4k**

Solution: **a . b** = (−6)*(12) + (3)*(0) + (−11)*(4)

= −72 + 0 − 44

= **−116**

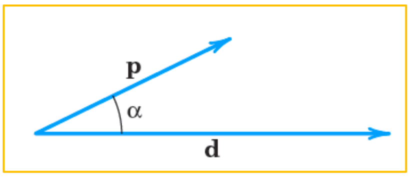
Note: Vector **a** and **b** are perpendicular to each other if and only if (**a . b** = 0) (Theorem1). And they are parallel if they are "multiples" of each other, like:

a = [2,4], **b** = [4,8], **c** = [1,2], and **d** = [-2,-4]. (These 2D vectors are all parallel, HOW!!)

Applications of Dot Product

WORK DONE BY A FORCE:

This is a major application of dot product. Let a constant force **P** acts on a body and makes a movement of the body by **d**, as shown, then the "work **W**" done is:



$$W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$$

Example#4: Find the work done by a force **P** acting on a body when it is displaced along a straight segment **AB** from A to B. Then find the angle **γ** between the force and the displacement. Knowing that **P** = [2,5,0], A = (1,3,3), and B = (3,5,5).

Solution: previously, we get the length of any vector is: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$

AB = B - A = (3,5,5) – (1,3,3) = [2,2,2]

The work done ***W*** = **P . AB** = [2,5,0] . [2,2,2] = (2*2+5*2+0*2) = **14**

$$\text{angle } \gamma = cos^{-1}\frac{\boldsymbol{P.AB}}{|\boldsymbol{P}||\boldsymbol{AB}|} = cos^{-1}\frac{14}{\sqrt{2^2+5^2+0^2}\,\sqrt{2^2+2^2+2^2}} = cos^{-1}\frac{14}{18.655} = \mathbf{41.4^o}$$

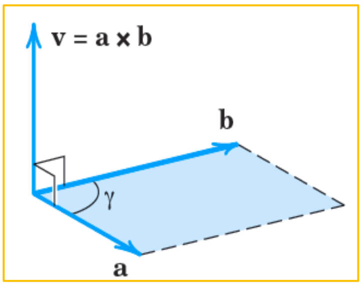
Note that the work done is +ve and the angle is < 90°

H. W.: Repeat Example#4 with **P** = [0,4,3], A = (4,5,-1), and B = (1,3,0).

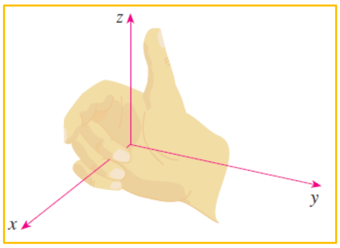
Answer: Work = -5, γ = 105.5°

Note that the work done is -ve and the angle is > 90°

Vector Product (Cross Product) of Two Vectors



We shall define another form of multiplication of vectors, whose result will be a *vector*. We can construct a vector **v** that is perpendicular to two vectors **a** and **b**, and the length of the resulting vector represents the area of the parallelogram containing the vectors **a** and **b**. The direction of **v** is determined by "**Right Hand Rule**" as shown. Therefore;

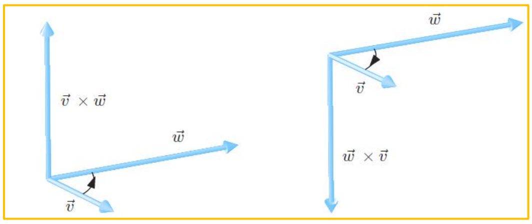


$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

Another form of cross product is:

$$\mathbf{a} \times \mathbf{b} = \mathbf{n}\,|\mathbf{a}||\mathbf{b}|\sin\gamma$$

Where **n** is a unit vector normal to both vectors **a** and **b**.



Basic Properties of Cross Product:

1) If **a** = 0 or **b** = 0, then **v** = **a** x **b** = 0
2) If both vectors are nonzero, then **v** has "length" $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\gamma$
3) The length of vector |**v**| represents the area of the parallelogram containing the multiplied vectors (**a** & **b**).
4) If **a** and **b** lie in the same straight line, then γ is 0^o or 180^o. Knowing that $\sin 0^o = 0$, and this gives **v = a x b = 0**

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

and $\mathbf{v} = [v_1, v_2, v_3] = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$

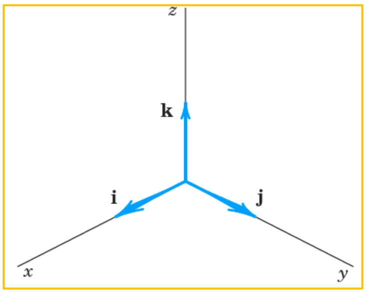
Example#1: Find the vector product **v** = **a** x **b** of **a** = [1,1,0], and **b** = [3,0,0].

Solution:

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}\mathbf{k} = -3\mathbf{k} = [0, 0, -3]$$

Knowing that;

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$



Example#2: Find the cross (vector) product of vectors **v** = 2i + j – 2k and **w** = 3i + k and show that the resulting vector is perpendicular to both **v** and **w** vectors.

Solution: Find **v** x **w** using second and third determinant;

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= i\big(1(1) - 0(-2)\big) - j\big(2(1) - 3(-2)\big) + k\big(2(0) - 3(1)\big) = \mathbf{i - 8j - 3k}$$

To show that this vector is perpendicular to both **v** and **w**, compute the dot product of the following;

$$\mathbf{v} \,.\, (\mathbf{v} \times \mathbf{w}) = (2i + j - 2k).(i - 8j - 3k) = 2 - 8 + 6 = 0$$

Similarly; $$\mathbf{w} \,.\, (\mathbf{v} \times \mathbf{w}) = (3i + 0j + k).(i - 8j - 3k) = 3 + 0 - 3 = 0$$

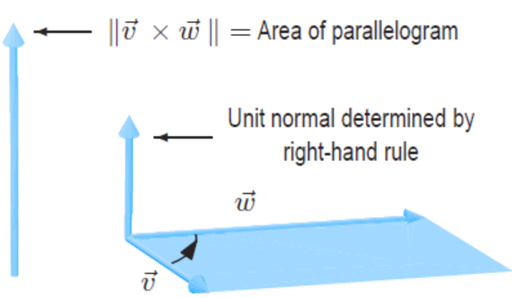
Thus; the vector resulting from **v** x **w** is perpendicular to both **v** and **w**, because zero dot product means normality.

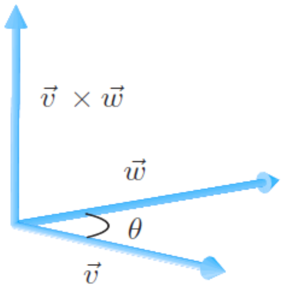
General Rules for Vector Product: If **a, b,** and **c** are vectors, and *l* is a scalar:

1) (*l***a**) x **b** = *l*(**a** x **b**) = **a** x (*l***b**)
2) **a** x (**b** + **c**) = (**a** x **b**) + (**a** x **c**)
3) (**a** + **b**) x **c** = (**a** x **c**) + (**b** x **c**)
4) **b** x **a** = - (**a** x **b**)
5) **a** x (**b** x **c**) ≠ (**a** x **b**) x **c**

Applications of Cross Product

AREA OF PARALLELOGRAM:





Example#1: Find the area of the parallelogram with edges **v** = 2i + j -3k and **w** = i + 3j +2k.

Solution: Using cross product;

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2+9)i - (4+3)j + (6-1)k = 11i - 7j + 5k$$

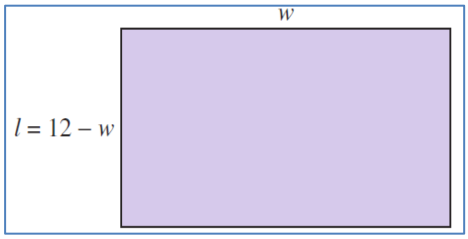
The area of parallelogram is; Area $= |\mathbf{v} \times \mathbf{w}| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}$

H.W.: **Given the points: P(1,1,1), Q(2,1,3), and R(3,-1,1). Find the area of the triangle determined by these three points.** **Ans.: Area = 3**

LIMITS

Example#1: If you are given 24 cm of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have?

Solution: Let w represent the width of the rectangle and let l represent the length of the rectangle. Because, $2w + 2l = 24$



Therefore, the area is $A = l * w = (12 - w)\, w = 12w - w^2$

Now, to obtain the maximum area we experiment different values of w, After trying several values, it appears that the maximum area occurs when, $w = 6$, as shown in table,

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

OR, you can say that "the limit of A as w approaches 6 is 36".

$$\lim_{w\to 6} A = \lim_{w\to 6}(12w - w^2) = 36$$

Definition of Limit

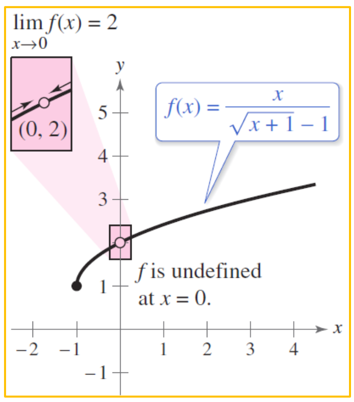
If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x\to c} f(x) = L.$$

Example#2: Given $f(x) = \frac{x}{\sqrt{x+1}-1}$, find the value of $f(x)$ at $x = 0$ using limit table.

Solution: substituting directly the value of $x = 0$ in the equation gives $0/0$, which is numerically undefined, but drawing the function shows a value at $x = 0$!!

So, we can construct a table that shows values of $f(x)$ for two sets of x-values, one approaches 0 from left and one from right.



x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99949	1.99995	?	2.00005	2.00050	2.00499

It appears that the limit is 2, which is also shown in figure.

Note that the function is not exist at $x = 0$, but the limit exist.

Therefore,

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

Example#3: Show that the limit is not exist for; $\lim_{x \to 0} \frac{|x|}{x}$

Solution:

Consider the graph of the function given by $f(x) = |x|/x$. In Figure , you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

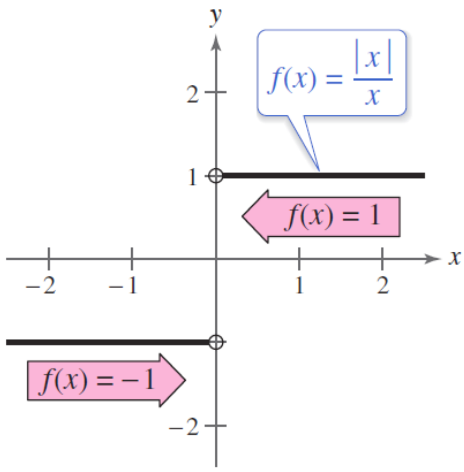
$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield

$$f(x) = 1$$

and

$$f(x) = -1.$$



This means that the limit is not exist.

The existence or nonexistence of f(x) at x=c has no effect on the existence of the limit of f(x) as x approaches c

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.

2. $f(x)$ increases or decreases without bound as x approaches c.

3. $f(x)$ oscillates between two fixed values as x approaches c.

Finding Limit using Direct Substitution

Direct substitution means: $\lim_{x \to c} f(x) = f(c)$. Substitute c for x.

Direct substitution is used to find the limit in the following **examples**:

a. $\lim_{x \to 4} x^2 = (4)^2 = 16$

b. $\lim_{x \to 4} 5x = 5 \lim_{x \to 4} x = 5(4) = 20$

c. $\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\lim_{x \to \pi} \tan x}{\lim_{x \to \pi} x} = \frac{0}{\pi} = 0$

d. $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$

e.
$$\begin{aligned} \lim_{x \to \pi} (x \cos x) &= \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) \\ &= \pi(\cos \pi) \\ &= -\pi \end{aligned}$$

f.
$$\begin{aligned} \lim_{x \to 3} (x + 4)^2 &= \left[\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 4\right)\right]^2 \\ &= (3 + 4)^2 \\ &= 7^2 = 49 \end{aligned}$$

g. $\lim_{x \to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$

h. $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$

Example#1: Find the limit; $\lim_{x \to -3} \frac{x^2+x-6}{x+3}$

Solution: If we substitute directly we get 0/0, therefore, algebraic treatment is needed:

$$\begin{aligned} \lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \\ &= \lim_{x \to -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} \\ &= \lim_{x \to -3} (x - 2) \\ &= -3 - 2 \\ &= -5 \end{aligned}$$

Example#2: Find the limit; $\lim_{x\to 1} \frac{x-1}{x^3-x^2+x-1}$

Solution:

$$\lim_{x\to 1} \frac{x-1}{x^3-x^2+x-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x^2+1)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x^2+1)}$$

$$= \lim_{x\to 1} \frac{1}{x^2+1}$$

$$= \frac{1}{1^2+1}$$

$$= \frac{1}{2}$$

Example#3: Find the limit of *f(x)* as *x* approaches 1.

$$f(x) = \begin{cases} 4-x, & x<1 \\ 4x-x^2, & x>1 \end{cases}$$

Solution:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (4-x)$$

$$= 4-1$$

$$= 3.$$

and,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x-x^2)$$

$$= 4(1)-1^2$$

$$= 3.$$

Therefore, the limit is exist, and $\boxed{\lim_{x\to 1} f(x) = 3.}$

DIFFERENTIATION

The derivative of a function at a point represents slope of the tangent for that curve at that point.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example#1: a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal -1/4?

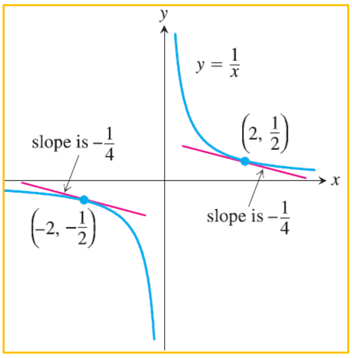
Solution: **(a)** Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-h}{ha(a+h)} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

When a = -1, the slope is $-1/(-1)^2$ = **-1**

(b) $-\frac{1}{a^2} = -\frac{1}{4}$ This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope -1/4 at the two points **(2, 1/2)** and **(-2, -1/2)**.



Now;

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Example#2: Using definition of derivative, differentiate $f(x) = \frac{x}{x-1}$

Solution: $f(x) = \frac{x}{x-1}$ and $f(x+h) = \frac{(x+h)}{(x+h)-1}$, so

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{Definition}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$$

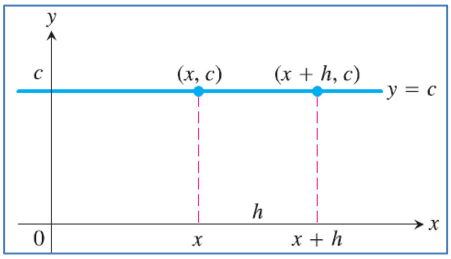
$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \qquad \text{Simplify}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \qquad \text{Cancel } h \neq 0$$

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$



Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Example#3: Differentiate the following powers of x;

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution: **(a)** $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ **(b)** $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ **(d)** $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example#4: Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Solution:
$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$
$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example#5: Find the derivative of $y = (x^2 + 1)(x^3 + 3)$

Solution: We can solve this example by two methods (a or b);

(a) $\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$ $\quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x.$$

(b) $y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Example#6: Find the derivative of $y = \frac{t^2-1}{t^3+1}$

Solution:

$$\frac{dy}{dt} = \frac{(t^3+1)\cdot 2t - (t^2-1)\cdot 3t^2}{(t^3+1)^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}$$

The second derivative is:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x)$$

OR generally, the n^{th} derivative is: $y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$

Example#7: Find all the derivatives of: $y = x^3 - 3x^2 + 2$

Solution:

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

Note: When we asked to find all the derivatives of a function, we stop when get 0.

Derivatives of Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}(\cos x) = -\sin x.$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

The Chain Rule:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

Example#1: Find the derivative of *g(t) = tan(5-sin2t)*

Solution:

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

Example#2: Find the derivative of the following functions:

(a) $(5x^3 - x^4)^7$ (b) $\frac{1}{3x-2}$ (c) $\sin^5 x$

Solution:

(a)
$$\begin{aligned} \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \\ &= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

Power Chain Rule with $u = 5x^3 - x^4, n = 7$

(b)
$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{3x - 2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\ &= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2) \\ &= -1(3x - 2)^{-2}(3) \\ &= -\frac{3}{(3x - 2)^2} \end{aligned}$$

Power Chain Rule with $u = 3x - 2, n = -1$

(c)
$$\begin{aligned} \frac{d}{dx}(\sin^5 x) &= 5\sin^4 x \cdot \frac{d}{dx}\sin x \\ &= 5\sin^4 x \cos x \end{aligned}$$

Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

Example3#: An object moves along the x-axis so that its position x at any time t is given by : $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution: We know that the velocity is dx/dt, $x = \cos(u)$ and $u = t^2+1$. We have:

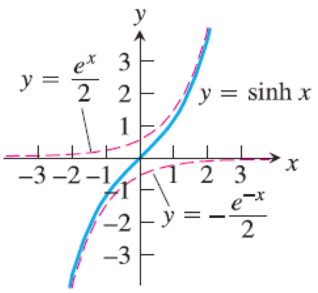
$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule;

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$
$$= -\sin(u) \cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u$$
$$= -\sin(t^2 + 1) \cdot 2t$$
$$= -2t \sin(t^2 + 1).$$

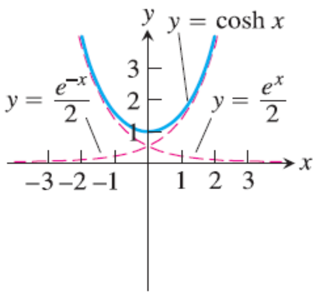
Hyperbolic Functions: Are functions formed by taking combinations of the two exponential functions (e^{x} and e^{-x}). The following are the basic six hyperbolic functions;



(a)

Hyperbolic sine:

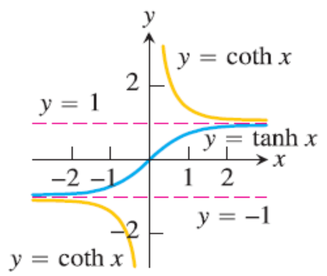
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



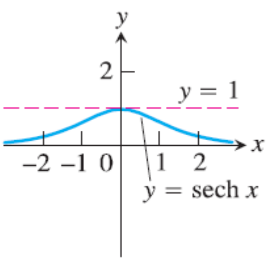
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

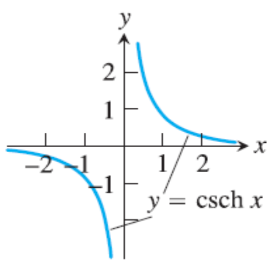
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Also, we have;

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives of Hyperbolic Functions:

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

L'Hopital's Rule: Is a method of differentiation to solve indeterminate limits. Indeterminant limits are limits of functions where both the numerator and the denominator are approaching 0 or positive or negative infinity.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$ **provided the limit exists**

Example#1: Evaluate the following limit: $\lim_{x\to 3} \frac{x^2-9}{x-3}$

Solution: The limits is indeterminate (0/0) when putting x = 3 !

The first method: factoring out (x-3) from the numerator, we get:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3} x+3 = 6$$

The second method: we can differentiate both the numerator and denominator according to L'Hopitals rule:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{2x}{1} = \lim_{x\to 3}(2*3) = 6$$

Note:* *We can differentiate more than one time

H.W. Using L'Hopital's rule, find $\lim_{x\to\infty} \frac{6x^2-4x}{3-5x^2}$ answer=6/-5

INVERSE FUNCTIONS

A function that undoes, or inverts, the effect of a function *f* is called the ***inverse*** of *f*.

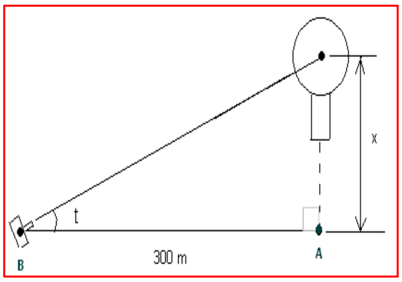
$$f^{-1}(b) = a \text{ if } f(a) = b.$$

Example#1: A camera is to take a series of photographs of a hot air balloon rising vertically. The distance between the camera at (B) and the launching point of the balloon (A) is 300 meters. The camera must keep the balloon on sight and therefore its angle of elevation t must change with the height x of the balloon.

a) Find angle t as a function of the height x.

b) Find angle t in degrees when x is equal to 150, 300 and 600 meters. (approximate your answer to 1 decimal place).

c) Graph t as a function of x.



Solution: tan(t) = x / 300

taking ($\tan^{-1}$) for the two sides; $\tan^{-1}(\tan(t)) = \tan^{-1}(x / 300)$

therefore, answer of branch (a) is $\mathbf{t = \tan^{-1}(x / 300)}$

(b) The values of t at 150, 300 and 600 are found using a calculator;

t(150) = 26.5 degrees (approximated to 1 decimal place)

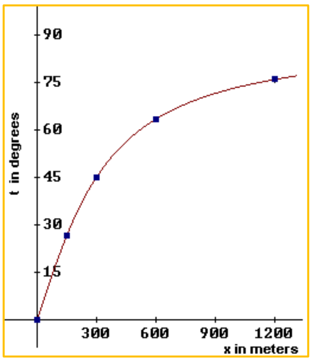
t(300) = 45.0 degrees

t(600) = 63.4 degrees (approximated to 1 decimal place)

(c) We use the values of t in part (b) and extra points and graph t as a function of x

OR, doing a table;

x	**t**
0	0
150	26.5
300	45.0
600	63.4
1200	76.0
3000	84.3



Derivative of Inverse Trigonometric Functions:

1. $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$

2. $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$

3. $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$

4. $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$

5. $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$

6. $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$

Example#2: Find the equation of the normal to the curve of $y = tan^{-1}(\frac{x}{2})\ \ at\ x = 3$

Solution: Benefit from: $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$

therefore, $\frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{2}\right)^2}\left(\frac{1}{2}\right)$

when $x = 3$, this expression is equal to: 0.153846, so the slope of the tangent at $x = 3$ is 0.153846. The slope of the normal at $x = 3$ is given by:

$\frac{-1}{0.153846} = -6.5$, so the **equation of the normal** is (when $x = 3$, $y = 0.9828$) given by:

$y - 0.9828 = -6.5(x - 3)$, OR $\boldsymbol{y = -6.5x + 20.483}$

NATURAL LOGARITHMS

In a simple form, a logarithm answers the question:

"How many of _one number_ do we multiply to get _another number_?"

i.e., Ex: How many 2's do we multiply to get 16?

Answer: 2*2*2*2 = 16. So we need to multiply 4 of the 2's to get 16

Now, we can say **"the logarithm of 16 with base 2 is 4"**, and it is written as:

$$\log_2(16) = 4$$

By the same thing, **"the logarithm of 10000 with base 10 is 4"** because; 10*10*10*10 = 10000, and written as: $\log_{10}(10000) = 4$

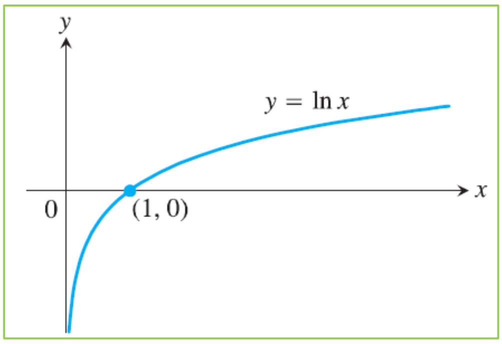
where as in natural logarithm (i.e., with base **e** "Euler's Number") gives the idea about how many times we need to multiply **e** to get the number. **e** = 2.718

$$\log_e(x) = \ln(x)$$

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$. For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1.. The function is not defined for $x \le 0$, also;



$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

DEFINITION The **number *e*** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1.$$

The derivative of $\ln x$ is:
$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

OR generally;
$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.$$

Example#1: Find the derivative of (a) $\ln 2x$ and (b) $\ln u$, where $u = (x^2 + 3)$.

Solution: (a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) $\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$

Properties of the Natural Logarithm; For any numbers $b > 0$ and $x > 0$;

1. *Product Rule*:	$\ln bx = \ln b + \ln x$
2. *Quotient Rule*:	$\ln\frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*:	$\ln\frac{1}{x} = -\ln x$
4. *Power Rule*:	$\ln x^r = r\ln x$

The following **examples** show the application of these properties;

(a) $\ln 4 + \ln\sin x = \ln(4\sin x)$ Product

(b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

(c) $\ln\frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3\ln 2$ Power

Also, If u is a differentiable function that is never zero,

$$\int\frac{1}{u}\,du = \ln|u| + C.$$

EXPONENTIAL FUNCTIONS

"Exponential function e^x is the inverse of $\ln x$."

"e (Euler's Number) is the x-value that gives $y = 1$ for the function $y = \ln x$."

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln(e^x) = x \qquad (\text{all } x)$$

Example#1: Solve the equation $e^{2x-6} = 4$ for x.

Solution: Taking the natural logarithm of both sides of the equation:

$$\begin{aligned} \ln\left(e^{2x-6}\right) &= \ln 4 \\ 2x - 6 &= \ln 4 \\ 2x &= 6 + \ln 4 \\ x &= 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{1/2} \\ x &= 3 + \ln 2 \end{aligned}$$

Properties of ln:

$$\begin{aligned} \ln\left(e^{x}\right) &= x \\ \frac{d}{dx}\ln\left(e^{x}\right) &= 1 \\ \frac{1}{e^{x}} \cdot \frac{d}{dx}\left(e^{x}\right) &= 1 \\ \frac{d}{dx}e^{x} &= e^{x}. \end{aligned}$$

If u is any differentiable function of x, then

$$\frac{d}{dx}e^{u} = e^{u}\frac{du}{dx}.$$

The following ***examples*** show the application of *ln* properties;

(a) $\frac{d}{dx}\left(5e^{x}\right) = 5\frac{d}{dx}e^{x} = 5e^{x}$

(b) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$

(d) $$\begin{aligned} \frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) &= e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right) \\ &= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}} \end{aligned}$$

Also,

The general antiderivative of the exponential function

$$\int e^{u}\,du = e^{u} + C$$

THEOREM For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^r = e^{rx_1}$, if r is rational

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives
For $x > 0$ and any real number n,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Example#2: Find equation of the slope (*dy/dx*) for the function:

$$y = 3x^{2.1} - 4\sin(2x) + 2e^{5x} + \frac{2}{x^2}$$

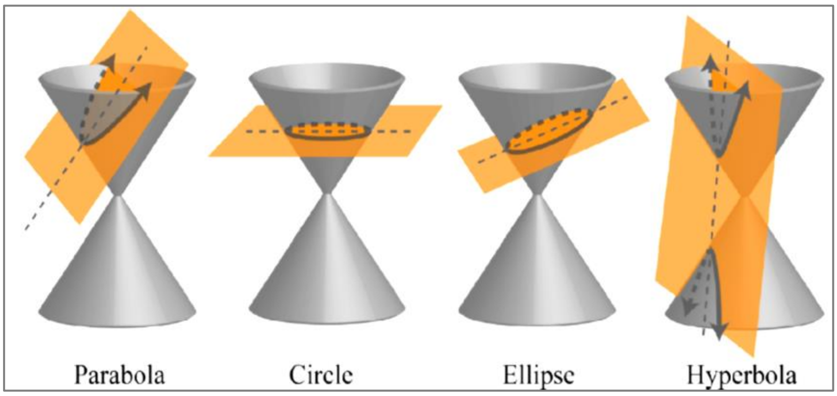
Solution:

$$dy = 3 * 2.1 * x^{2.1-1}\, dx - 4\cos(2x) * 2dx + 2 * e^{5x} * 5dx + 2 * (-2)(x^{-2-1})dx$$

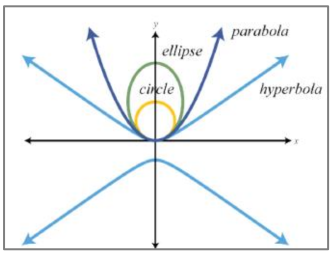
The equation of slope $\frac{dy}{dx} = 6.3x^{1.1} - 8\cos(2x) + 10e^{5x} - \frac{4}{x^3}$

CONIC SECTIONS

A **conic section** is a curve obtained from the intersection of a right circular cone and a plane. There are four conic sections: parabola, circle, ellipse, and hyperbola.



The goal is to sketch these graphs on a rectangular coordinate plane (x and y), as shown below;



First we began with the **Distance Formula**: Given two points (x_1 , y_1) and (x_2 , y_2) in a rectangular coordinate plane, the distance *d* between them is given by the distance formula;

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

And the **midpoint** that divides this distance *d* into two equal parts has the coordinates;

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example#1: Given (−2, −5) and (−4, −3) calculate the distance and midpoint between these two points.

Solution:

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[-4 - (-2)]^2 + [-3 - (-5)]^2} \\ &= \sqrt{(-4 + 2)^2 + (-3 + 5)^2} \\ &= \sqrt{(-2)^2 + (2)^2} \\ &= \sqrt{4 + 4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

And the midpoint has the coordinates;

$$\begin{aligned} \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) &= \left(\frac{-2 + (-4)}{2}, \frac{-5 + (-3)}{2}\right) \\ &= \left(\frac{-6}{2}, \frac{-8}{2}\right) \\ &= (-3, -4) \end{aligned}$$

Example#2: The diameter of a circle is defined by the two points (−1, 2) and (1, −2). Determine the radius of the circle and use it to calculate its area.

Solution: Find the diameter using the distance formula;

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{[1 - (-1)]^2 + (-2 - 2)^2} \\
&= \sqrt{(2)^2 + (-4)^2} \\
&= \sqrt{4 + 16} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}
$$

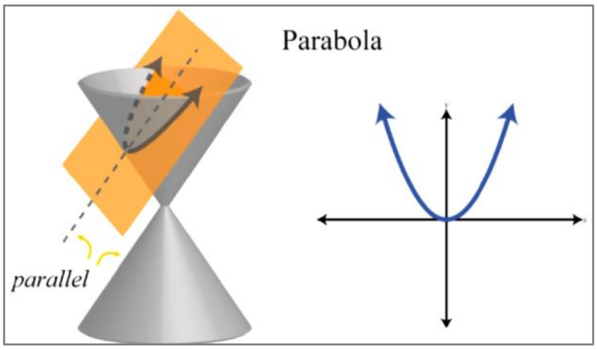
And radius of the circle r is; $r = \frac{d}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$

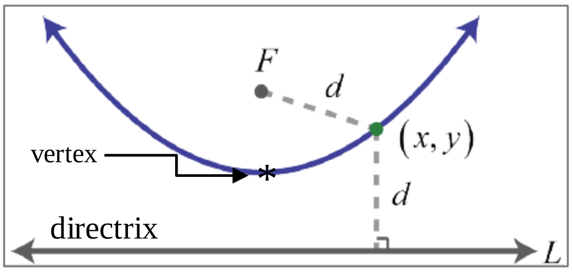
The area of a circle is given by the formula $A = \pi r^2$ and we have; $A = \pi\left(\sqrt{5}\right)^2$

$$
\begin{aligned}
&= \pi \cdot 5 \\
&= 5\pi
\end{aligned}
$$

1) The Parabola:

A **parabola** is the curve formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone. Also, the **parabola** is the set of points (x,y) in a plane equidistant (d) from a given line (L), called the directrix, and a point not on the line, called the focus (F), as shown;





The **vertex** of the parabola is the point where the shortest distance to the directrix is at a minimum.

Now, let the graph of a quadratic function, a polynomial function of degree 2, is parabolic. We can write the equation of a parabola either in **general form** or in **standard form**:

General Form	Standard Form
$y = ax^2 + bx + c$	$y = a(x-h)^2 + k$

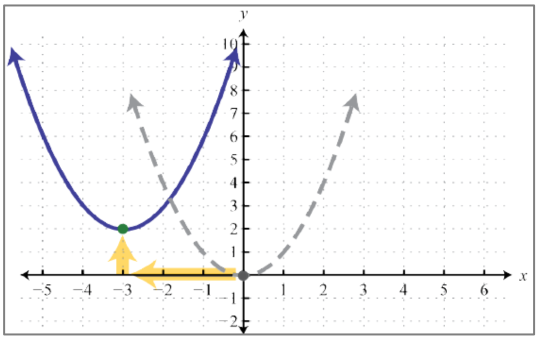
Note: Here point (h,k) represents the **vertex**, therefore, the standard form is important.

For Example; to sketch the parabola: $y = (x+3)^2 + 2$;

$y = x^2$ *Basic squaring function.*

$y = (x+3)^2$ *Horizontal shift left 3 units.*

$y = (x+3)^2 + 2$ *Vertical shift up 2 units.*



Here we can see that the vertex is $(-3, 2)$. This can be determined directly from the equation in standard form;

$$y = a(x-h)^2 + k$$
$$\downarrow \quad \downarrow$$
$$y = [x-(-3)]^2 + 2$$

Therefore, it is important to know how to transform from general form to standard form by "***completing the square***" method, as shown in the following example;

Example#1: Rewrite the equation in standard form and determine the vertex of the graph: $y = x^2 - 8x + 15$

Solution: The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$.

In this case, $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$, then;

$$\begin{aligned} y &= x^2 - 8x + 15 \\ &= \left(x^2 - 8x + 16\right) + 15 - 16 \\ &= (x-4)(x-4) - 1 \\ &= (x-4)^2 - 1 \end{aligned}$$

Note that to complete the square, x^2 factor must be 1.

So,

$$y = a(x-h)^2 + k$$
$$\downarrow \quad \downarrow$$
$$y = (x-4)^2 + (-1)$$

therefore, the vertex is (4,-1).

Example#2: Rewrite the equation in standard form and determine the vertex of the graph: $y = -2x^2 + 12x - 16$, then sketch this function.

Solution: Since $a = -2$, factor this out of the first two terms in order to complete the square;

$$y = -2x^2 + 12x - 16$$
$$= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16$$

Now use −6 to determine the value that completes the square. In this case,

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now,

$$\begin{aligned} y &= -2x^2 + 12x - 16 \\ &= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16 \\ &= -2(x^2 - 6x + 9 - 9) - 16 \\ &= -2\left[(x-3)(x-3) - 9\right] - 16 \\ &= -2\left[(x-3)^2 - 9\right] - 16 \\ &= -2(x-3)^2 + 18 - 16 \\ &= -2(x-3)^2 + 2 \end{aligned}$$

So, the vertex form: $y = a(x-h)^2 + k$

↓ ↓

$$y = -2(x-3)^2 + 2$$

and the vertex is (3,2).

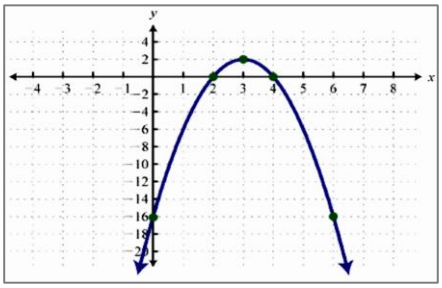
Now, to graph this function, we have the two forms:

General Form	Standard Form
$y = -2x^2 + 12x - 16$	$y = -2(x-3)^2 + 2$

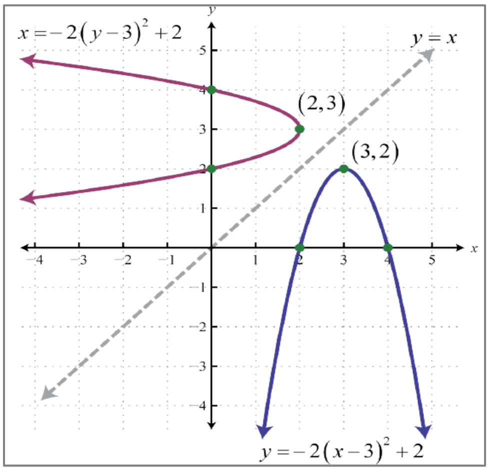
Note that if the coefficient $a > 0$ the parabola opens upward and if $a < 0$ the parabola opens downward. In this case, $a = -2$ and we conclude the parabola opens downward. Use general form to determine the y-intercept. When $x = 0$ we can see that the y-intercept is (0, −16). From the equation in standard form, we can see that the vertex is (3, 2) .To find the x-intercept we could use either form. In this case, we will use standard form to determine the x-values where $y = 0$,

$$y = -2(x-3)^2 + 2 \qquad \textit{Set } y = 0 \textit{ and solve.}$$
$$0 = -2(x-3)^2 + 2$$
$$-2 = -2(x-3)^2$$
$$1 = (x-3)^2 \qquad \textit{Apply the square root property.}$$
$$\pm 1 = x - 3$$
$$3 \pm 1 = x$$

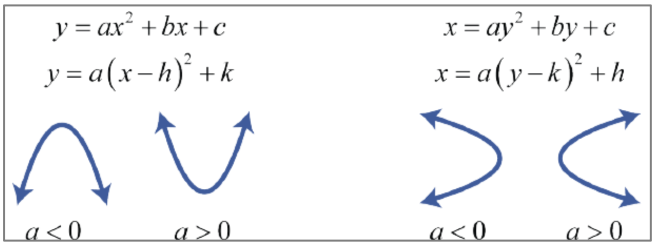
Here $x = 3 - 1 = 2$ or $x = 3 + 1 = 4$ and therefore the x-intercepts are (2, 0) and (4, 0). Use this information to sketch the graph,



We can extend our study to include parabolas that open right or left. If we take the equation that defines the parabola in the previous example, $y = -2(x-3)^2 + 2$, and switch the x and y values we obtain: $x = -2(y-3)^2 + 2$. This produces a new graph with symmetry about the line $y = x$, as shown below;



and, in general,



In all cases, the vertex is (h, k). Take care to note the placement of h and k in each equation.

Example#3: Graph: $x = y^2 + 10y + 13$ after finding its vertex.

Solution: Because the coefficient of y^2 is positive, $a = 1$, we conclude that the graph is a parabola that opens to the right. Furthermore, when $y = 0$ it is clear that $x = 13$ and therefore the x-intercept is $(13, 0)$. Complete the square to obtain standard form. Here we will add and subtract: $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$

$$\begin{aligned} x &= y^2 + 10y + 13 \\ &= y^2 + 10y + 25 - 25 + 13 \\ &= (y+5)(y+5) - 12 \\ &= (y+5)^2 - 12 \end{aligned}$$

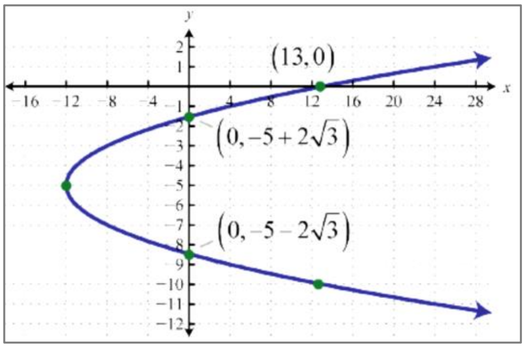
Thus, to find the vertex:

$$x = a\,(y-k)^2 + h$$
$$\downarrow \qquad \downarrow$$
$$x = (y - (-5))^2 + (-12)$$

From this we can see that the vertex $(h, k) = (-12, -5)$ .Next use standard form to find the y-intercepts by setting $x = 0$.

$$\begin{aligned} x &= (y+5)^2 - 12 \\ 0 &= (y+5)^2 - 12 \\ 12 &= (y+5)^2 \\ \pm\sqrt{12} &= y+5 \\ \pm 2\sqrt{3} &= y+5 \\ -5 \pm 2\sqrt{3} &= y \end{aligned}$$

Or y = -1.5 and -8.5. Thus, the y-intercepts is (0,-1.5) and (0,-8.5). The graph will be:

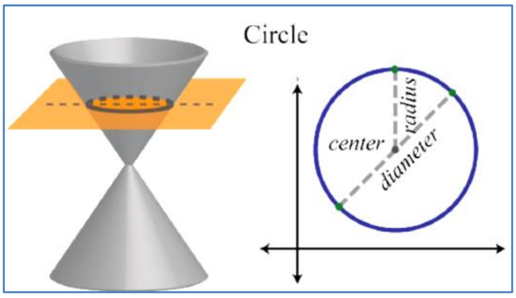


H.W.#1: Find the vertex and graph the function: $x = -2y^2 + 4y - 5$ **Ans: the vertex is (-3,1).**

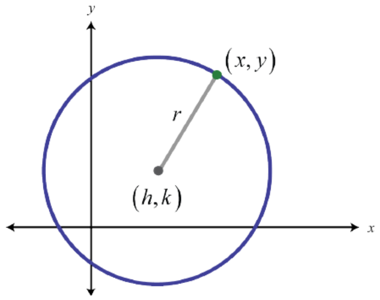
H.W.#2: Find the vertex and graph the function: $x = y^2 - y - 6$. **Ans: the vertex is (-6.25,0.5).**

2) The Circle:

A **circle** is the set of points in a plane that lie a fixed distance, called the **radius**, from any point, called the **center**. The **diameter** is the length of a line segment passing through the center whose endpoints are on the circle. In addition, a circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone:



In a rectangular coordinate plane, where the center of a circle with radius r is (h, k), we have:



Calculate the distance between (h, k) and (x, y) using the distance formula:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Squaring both sides leads us to the equation of a circle in **Standard Form**;

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{(Standard Form)}$$

In this form, the center and radius are apparent. For example, given the equation:

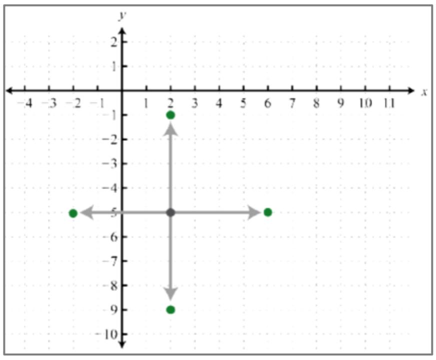
$(x-2)^2 + (y+5)^2 = 16$ we have,

$$\begin{array}{ccc} (x-h)^2 + & (y-k)^2 & = r^2 \\ \downarrow & \downarrow & \downarrow \\ (x-2)^2 + & [y-(-5)]^2 & = 4^2 \end{array}$$

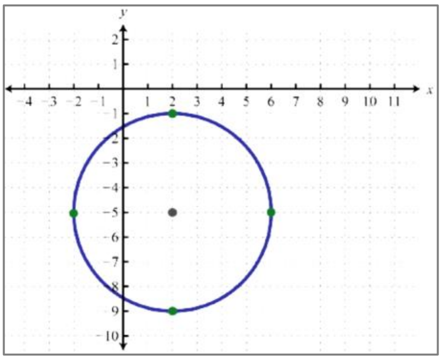
In this case, the center is $(2, -5)$ and $r = 4$. The graph of a circle is determined by its center and radius.

Example#1: Graph: $(x-2)^2 + (y+5)^2 = 16$, then find the x and y intercepts.

Solution: Written in this form we can see that the center is $(2, -5)$ and that the radius $r = 4$ units. From the center mark points 4 units up and down as well as 4 units left and right, then draw the circle through these four points, as shown;



(a)



(b)

To find the y intercepts set $x = 0$:

$$\begin{aligned}(x-2)^2 + (y+5)^2 &= 16\\(0-2)^2 + (y+5)^2 &= 16\\4 + (y+5)^2 &= 16\\(y+5)^2 &= 12\\y+5 &= \pm\sqrt{12}\\y+5 &= \pm 2\sqrt{3}\\y &= -5 \pm 2\sqrt{3}\end{aligned}$$

Or y = -1.5 and -8.5. Thus, the y-intercepts are (0,-1.5) and (0,-8.5).

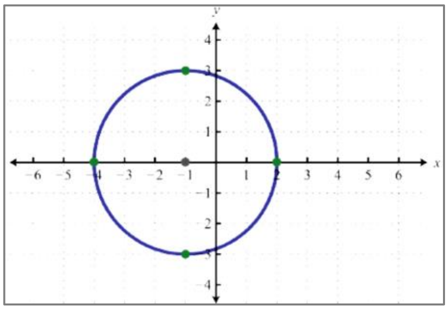
To find the x-intercepts set $y = 0$:

$$\begin{aligned}(x-2)^2 + (y+5)^2 &= 16\\(x-2)^2 + (0+5)^2 &= 16\\(x-2)^2 + 25 &= 16\\(x-2)^2 &= -9\\x-2 &= \pm\sqrt{-9}\\x &= 2 \pm 3i\end{aligned}$$

and because the solutions are complex we conclude that there are no real x-intercepts.

Example#2: Graph the circle with radius $r = 3$ units centered at $(-1, 0)$. Then give its equation in standard form and determine the intercepts.

Solution: Given that the center is $(-1, 0)$ and the radius is $r = 3$ we sketch the graph as follows:



Substitute h, k, and r to find the equation in standard form. Since $(h, k) = (-1, 0)$and $r = 3$ we have,

$$(x-h)^2 + (y-k)^2 = r^2$$

$$[x-(-1)]^2 + (y-0)^2 = 3^2$$

$$(x+1)^2 + y^2 = 9$$

The equation of the circle is $(x + 1)^2 + y^2 = 9$, use this to determine the y-intercepts.

Setting $x = 0$ gives:

$$(x+1)^2 + y^2 = 9$$

$$(0+1)^2 + y^2 = 9$$

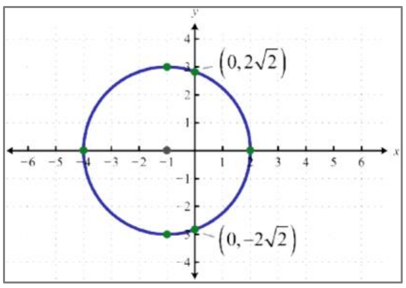
$$1 + y^2 = 9$$

$$y^2 = 8$$

$$y = \pm\sqrt{8}$$

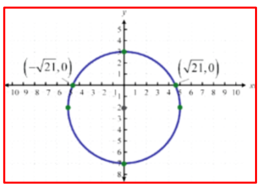
$$y = \pm 2\sqrt{2}$$

Therefore, the y-intercepts are $(0, 2\sqrt{2})$and $(0, -2\sqrt{2})$. Then set $y = 0$ and solve for x; gives x-intercepts (-4,0) and (2,0) (Solve it as a H.W.). The graph will be as shown:



H.W.#1: Graph the circle and set the intercepts for: $x^2 + (y + 2)^2 = 25$.

Answer:



We have seen that the graph of a circle is completely determined by the center and radius which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a circle in **General Form** is:

$$x^2 + y^2 + cx + dy + e = 0$$

Here c, d, and e are real numbers. The steps for graphing a circle given its equation in general form is as in the following example;

Example#3: Graph: $x^2 + y^2 + 6x - 8y + 13 = 0$.

Solution: Begin by rewriting the equation in standard form;

$$x^2 + y^2 + 6x - 8y + 13 = 0$$

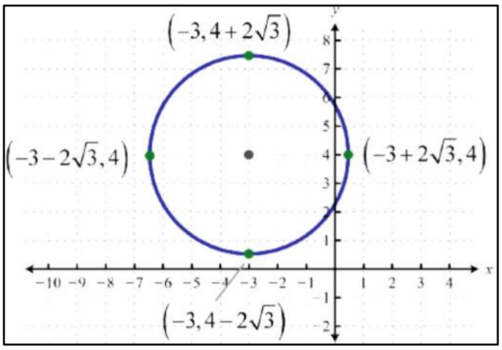
$$(x^2 + 6x + \_\_\_) + (y^2 - 8y + \_\_\_) = -13$$

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = -13 + 9 + 16$$

Then,

$$(x+3)^2 + (y-4)^2 = 12$$

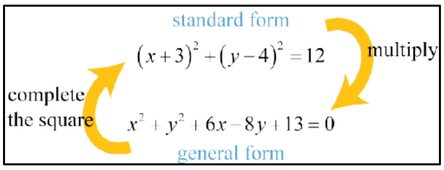
So, the center is (-3,4), and the radius is $\sqrt{12} = 3.46$. Then mark the radius vertically and horizontally and then sketch the circle through these points;



H.W.#2: Determine the center and radius for: $4x^2 + 4y^2 - 8x + 12y - 3 = 0$.

Hint: Firstly divide the two sides by 4. Answer: Center (1, -1.5), $r = 2$

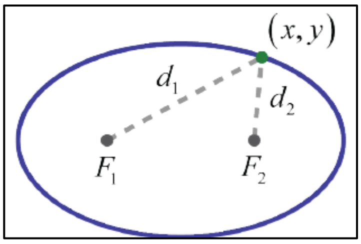
In summary, to convert from standard form to general form we multiply, and to convert from general form to standard form we complete the square.



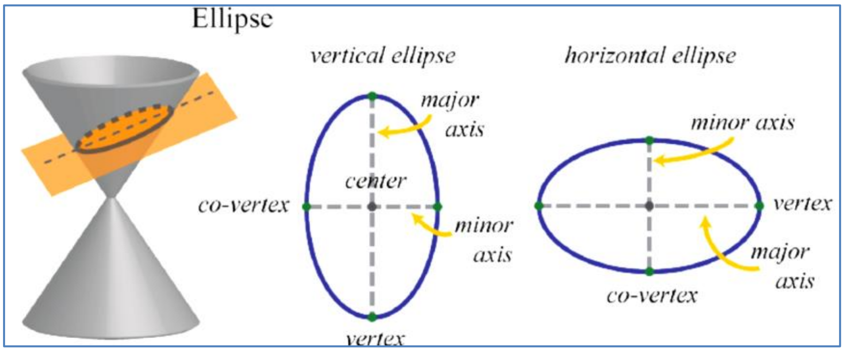
3) The Ellipse:

An **ellipse** is the set of points (x,y) in a plane whose distances (d_1, d_2) from two fixed points (F_1, F_2) called foci, have a sum that is equal to a positive constant (d).

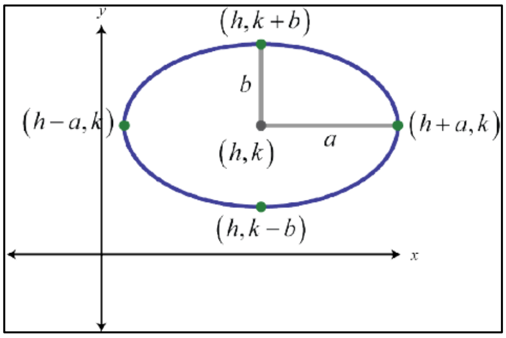
$$d = d_1 + d_2$$



In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone.



If the major axis of an ellipse is parallel to the x-axis in a rectangular coordinate plane, we say that the ellipse is horizontal. If the major axis is parallel to the y-axis, we say that the ellipse is vertical. In our study, we are only concerned with sketching these two types of ellipses. In a rectangular coordinate plane, where the center of a horizontal ellipse is (h, k), we have;



Where a is called the major radius and b is the minor radius. The equation of the ellipse in **Standard From** is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The vertices are $(h \pm a, k)$ and $(h, k \pm b)$ and the orientation depends on a and b. If $a > b$, then the ellipse is horizontal as shown above and if $a < b$, then the ellipse is vertical and b becomes the major radius. What do you think happens when $a = b$?

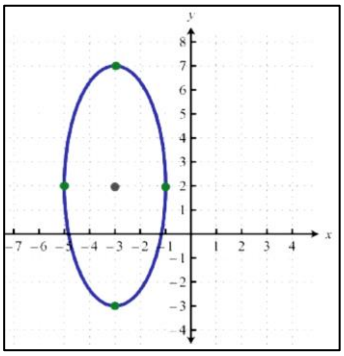
The following table gives examples on ellipse;

Equation	Center	a	b	Orientation
$\frac{(x-1)^2}{4}+\frac{(y-8)^2}{9}=1$	$(1,8)$	$a=2$	$b=3$	Vertical
$\frac{(x-3)^2}{2}+\frac{(y+5)^2}{16}=1$	$(3,-5)$	$a=\sqrt{2}$	$b=4$	Vertical
$\frac{(x+1)^2}{1}+\frac{(y-7)^2}{8}=1$	$(-1,7)$	$a=1$	$b=2\sqrt{2}$	Vertical
$\frac{x^2}{25}+\frac{(y+6)^2}{10}=1$	$(0,-6)$	$a=5$	$b=\sqrt{10}$	Horizontal

The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius, all of which can be determined from its equation written in standard from.

Example#1: Graph: $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{25}=1$, then find the intercepts.

Solution: Written in this form we can see that the center of the ellipse is (-3,2), $a=\sqrt{4}=2, b=\sqrt{25}=5$, therefore the graph is as shown;



Now, to find the x-intercepts, set $y = 0$;

$$\frac{(x+3)^2}{4}+\frac{(0-2)^2}{25}=1$$

$$\frac{(x+3)^2}{4}+\frac{4}{25}=1$$

$$\frac{(x+3)^2}{4}=1-\frac{4}{25}$$

$$\frac{(x+3)^2}{4}=\frac{21}{25}$$

Taking square root for both sides, we get; $\frac{x+3}{2} = \pm\sqrt{\frac{21}{25}}$

$$x + 3 = \pm \frac{2\sqrt{21}}{5}$$

$$x = -3 \pm \frac{2\sqrt{21}}{5} = \frac{-15 \pm 2\sqrt{21}}{5}$$

x_1 = -1.17, x_2 = -4.83, and the x-intercepts are: (-1.17, 0) and (-4.83,0).

Setting x = 0 and solving for y leads to complex solutions, therefore, there are no y-intercepts. This is left as an exercise.

Example#2: Graph: $(x-2)^2 + 9(y-1)^2 = 9$, then find the intercepts.

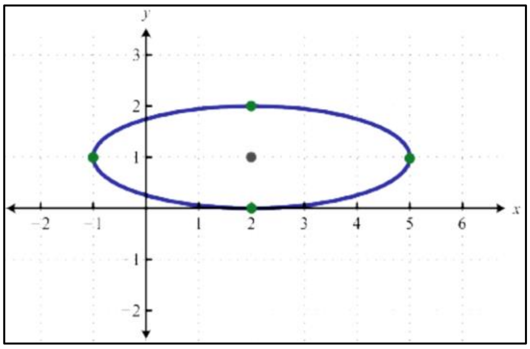
Solution: To obtain the standard form, with 1 on the right side, divide both sides by 9;

$$\frac{(x-2)^2 + 9(y-1)^2}{9} = \frac{9}{9}$$

$$\frac{(x-2)^2}{9} + \frac{9(y-1)^2}{9} = \frac{9}{9}$$

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

Therefore, the center of the ellipse is (2,1), $a = \sqrt{9} = 3, b = \sqrt{1} = 1$. The graph is:



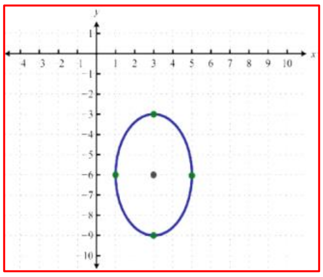
To find the intercepts we can use the standard form $\frac{(x-2)^2}{9} + (y-1)^2 = 1$;

x-intercepts set $y=0$	y-intercepts set $x=0$
$\frac{(x-2)^2}{9}+(0-1)^2=1$ $\frac{(x-2)^2}{9}+1=1$ $(x-2)^2=0$ $x-2=0$ $x=2$	$\frac{(0-2)^2}{9}+(y-1)^2=1$ $\frac{4}{9}+(y-1)^2=1$ $(y-1)^2=\frac{5}{9}$ $y-1=\pm\sqrt{\frac{5}{9}}$ $y=1\pm\frac{\sqrt{5}}{3}=\frac{3\pm\sqrt{5}}{3}$ = **1.75** and **0.25**

Therefore, the x-intercept is (2,0), and the y-intercepts are (0,1.75) and (0,0.25).

H.W.#1: Graph: $9(x-3)^2+4(y+2)^2=36$

Answer:



Equation of ellipse in General Form is: $px^2+qy^2+cx+dy+e=0$

where $p, q>0$. The steps for graphing an ellipse given its equation in general form are outlined in the following example;

Example#3: Graph: $2x^2+9y^2+16x-90y+239=0$, then find the intercepts.

Solution: Begin by rewriting the equation in standard form;

$$2x^2+9y^2+16x-90y+239=0$$
$$(2x^2+16x+\_\_\_)+(9y^2-90y+\_\_\_)=-239$$
$$2(x^2+8x+\_\_\_)+9(y^2-10y+\_\_\_)=-239$$

Now, to complete the squares, for the x-terms use: $\left(\frac{8}{2}\right)^2=4^2=16$, and for y-terms use: $\left(\frac{-10}{2}\right)^2=(-5)^2=25$, therefore;

$$2(x^2+8x+16)+9(y^2-10y+25)=-239+32+225$$

Divide to get 1 on the right side and simplify;

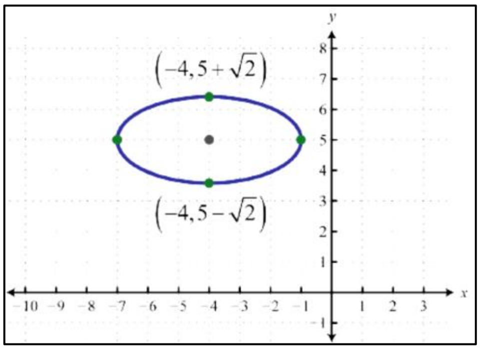
$$2(x+4)^2+9(y-5)^2=18$$

$$\frac{2(x+4)^2+9(y-5)^2}{18}=\frac{18}{18}$$

$$\frac{2(x+4)^2}{18}+\frac{9(y-5)^2}{18}=\frac{18}{18}$$

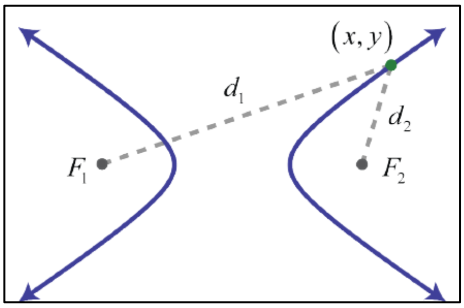
$$\frac{(x+4)^2}{9}+\frac{(y-5)^2}{2}=1$$

Therefore, the center is (-4,5), $a=\sqrt{9}=3$, and $b=\sqrt{2}$, and the graph will be:



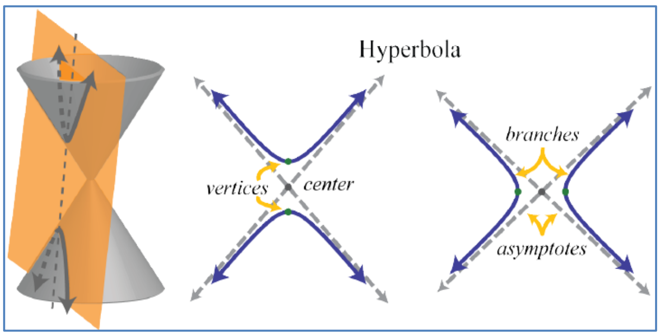
4) The Hyperbola:

A **hyperbola** is the set of points (*x*,*y*) in a plane whose distances (d_1 and d_2) from two fixed points, (F_1 and F_2) called foci, has an absolute difference that is equal to a positive constant (*d*). Therefore, $\boldsymbol{d=|d_1-d_2|}$



In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called **branches.**

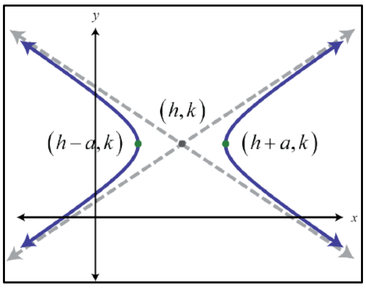
Points on the separate branches of the graph where the distance is at a minimum are called **vertices**. The midpoint between a hyperbola's vertices is its **center**.



Note: In this study, we will focus only on graphing hyperbolas that open left and right or upward and downward.

The equation of a hyperbola opening **left and right** in **standard form** is:

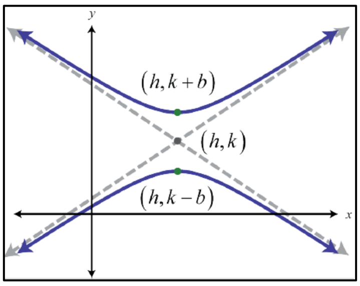
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



Here the center is (h,k) and the vertices are $(h+a,k)$ and $(h-a,k)$.

The equation of a hyperbola opening **upward and downward** in **standard form** is:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



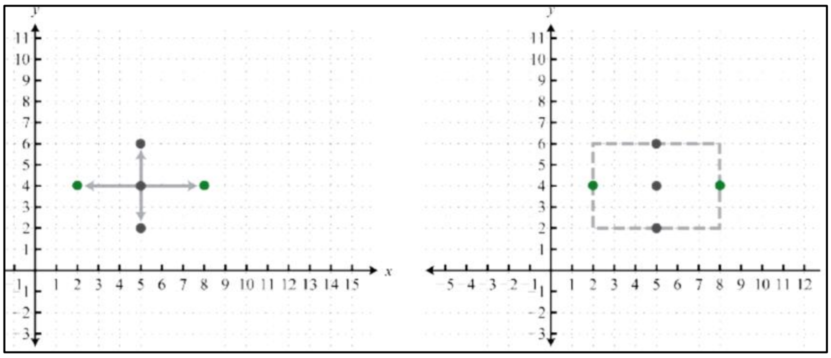
Here the center is (h,k) and the vertices are $(h,k+b)$ and $(h,k-b)$.

For example:

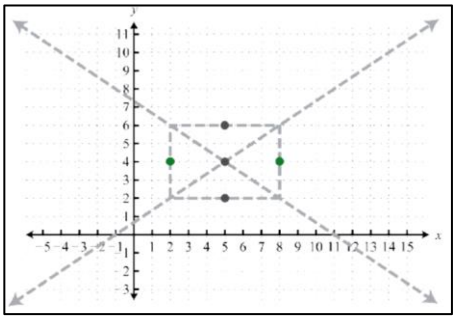
Equation	Center	a	b	Opens
$\frac{(x-3)^2}{25} - \frac{(y-5)^2}{16} = 1$	$(3,5)$	$a=5$	$b=4$	Left and right
$\frac{(y-2)^2}{36} - \frac{(x+1)^2}{9} = 1$	$(-1,2)$	$a=3$	$b=6$	Upward and downward
$\frac{(y+2)^2}{3} - (x-5)^2 = 1$	$(5,-2)$	$a=1$	$b=\sqrt{3}$	Upward and downward
$\frac{x^2}{49} - \frac{(y+4)^2}{8} = 1$	$(0,-4)$	$a=7$	$b=2\sqrt{2}$	Left and right

Example#1: Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1.$

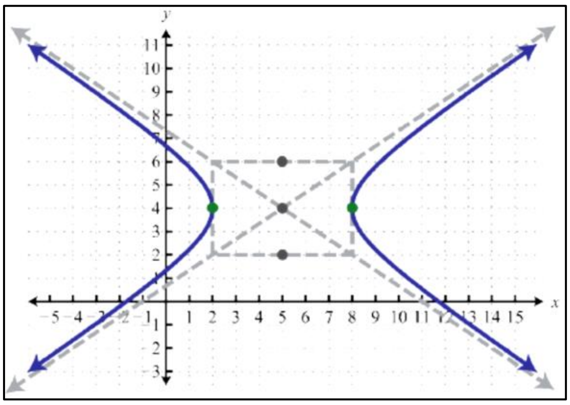
Solution: The coefficient of x is positive, therefore, the hyperbola opens left and right. Here $a = \sqrt{9} = 3$ *and* $b = \sqrt{4} = 2$. Now from the center (5,4):



The lines through the corners of this rectangle define the asymptotes.

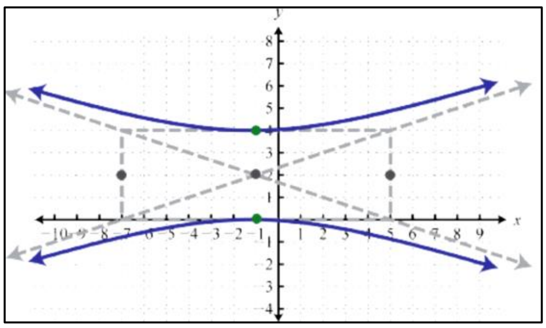


Use these dashed lines as a guide to graph the hyperbola opening left and right passing through the vertices;



Example#2: Graph: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$, then find the intercepts.

Solution: The coefficient of y is positive, therefore, the hyperbola opens upward and downward. Here $a = \sqrt{36} = 6$ *and* $b = \sqrt{4} = 2$. Now from the center (-1,2) and following the same steps in the previous example, the graph will be as shown:



Now, to find the x-intercepts set y = 0 and solve for x;

$$\frac{(0-2)^2}{4}-\frac{(x+1)^2}{36}=1$$

$$1-\frac{(x+1)^2}{36}=1$$

$$-\frac{(x+1)^2}{36}=0$$

$$(x+1)^2=0$$

$$x+1=0$$

$$x=-1$$

Therefore there is only one x-intercept (−1, 0). To find the y-intercept set x = 0 and solve for y;

$$\frac{(y-2)^2}{4}-\frac{(0+1)^2}{36}=1$$

$$\frac{(y-2)^2}{4}-\frac{1}{36}=1$$

$$\frac{(y-2)^2}{4}=\frac{37}{36}$$

$$\frac{(y-2)}{2}=\pm\frac{\sqrt{37}}{6}$$

$$y-2=\pm\frac{\sqrt{37}}{3}$$

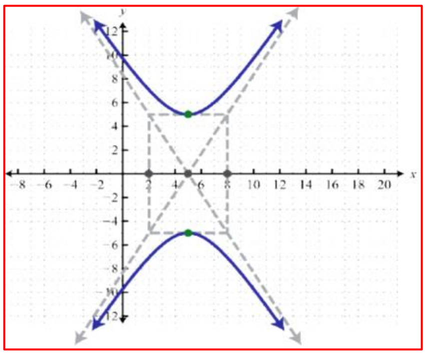
$$y=2\pm\frac{\sqrt{37}}{3}=\frac{6\pm\sqrt{37}}{3}$$

OR y_1 = -0.03, y_2 = 4.03

Therefore, there are two y-intercepts: (0,-0.03) and (0,4.03).

H.W.#1: Graph: $\frac{y^2}{25}-\frac{(x-5)^2}{9}=1$

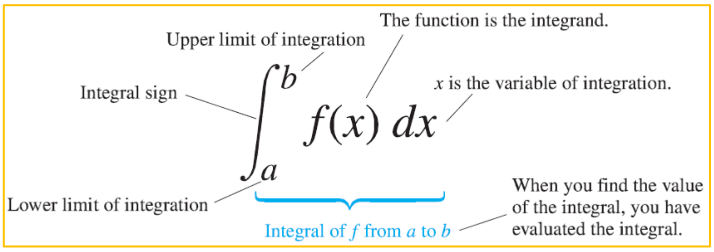
Answer:



INTEGRATION

OR: Anti-derivative

Definite Integration



Definite integral gives a <u>value</u>

Indefinite Integration

$$\int f(x)\,dx = F(x) + C$$

where C is constant

Indefinite integral gives an <u>equation</u>

<u>Rules of Definite Integration:</u> (Also applied to indefinite integration)

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

<u>INTEGRATION TECHNIQUES</u>

1) <u>*Substitution Method*</u>

THEOREM —The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

The Integrals of tan x, cot x, sec x, and csc x

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \qquad u = \sin x,\ du = \cos x\,dx$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

To integrate $\sec x$, we multiply and divide by $(\sec x + \tan x)$.

$$\int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \qquad u = \sec x + \tan x,\ du = (\sec x \tan x + \sec^2 x)\,dx$$

For $\csc x$, we multiply and divide by $(\csc x + \cot x)$.

$$\int \csc x\,dx = \int \csc x\,\frac{(\csc x + \cot x)}{(\csc x + \cot x)}\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \qquad u = \csc x + \cot x,\ du = (-\csc x \cot x - \csc^2 x)\,dx$$

SUMMARY

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Example#1: Find the value of the following definite integrals:

$(a)\ \int_0^{\pi} \cos x\,dx \qquad (b)\ \int_{-\pi/4}^{0} \sec x \tan x\,dx \qquad (c)\ \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

Solution:

(a) $\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi}$ because $\frac{d}{dx}\sin x = \cos x$

$$= \sin\pi - \sin 0 = 0 - 0 = 0$$

(b) $\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{0}$ because $\frac{d}{dx}\sec x = \sec x \tan x$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$ because $\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$

$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$

$$= [8 + 1] - [5] = 4.$$

$\because d(x^n) = nx^{n-1}$, integrate both sides: $\int d(x^n) = n\int x^{n-1}$, OR: $x^n = n\int x^{n-1}$

Therefore; $\int x^{n-1} = \frac{x^n}{n}$, OR: $\int x^n = \frac{x^{n+1}}{n+1}$, like: $\int x^2 = \frac{x^3}{3}$

Example#2: Find the integral $\int (x^3 + x)^5\,(3x^2 + 1)dx$

Solution: We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

Example#3: Benefit from the *ln* definition, find; $\int_0^2 \frac{2x}{x^2-5}dx$

Solution: Let $u = x^2 - 5$, gives $du = 2x\,dx$

so, $u(0) = -5$, and $u(2) = -1$

$$\int_0^2 \frac{2x}{x^2 - 5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5$$

$$\mathbf{= -\ln 5}$$

Example#4: Find (a) $\int_0^{\ln 2} e^{3x}dx,\quad and\ (b)\ \int_0^{\pi/2} e^{\sin x}\cos x\ dx$

Solution:

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du \qquad u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,\ u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$$

$$= e^1 - e^0 = e - 1$$

2) Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

Note that we try to choose u the function which may be disappeared by differentiation.

Example#1: Find $\int x\cos x\,dx$ using integration by parts

Solution: We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

Example#2: Find $\int \ln x\,dx$

Solution: Since $\int \ln x\,dx$ can be written as $\int \ln x.1\,dx$, we use the formula of by part;

$\int u\,dv = uv - \int v\,du$ with; $u = \ln x$, $du = dx/x$, $dv = 1dx$, $v = x$

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Example#3: Evaluate $\int x^2 e^x\,dx$

Solution: With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$$
$$= x^2 e^x - 2xe^x + 2e^x + C.$$

H.W.: Determine the following integral: $J = \int e^x \sin x\,dx$

---

3) Trigonometric Integrals

We begin with integrals of the form: $\int \sin^m x \cos^n x\,dx$, where m and n are nonnegative integers (+ve or 0). We can divide the appropriate substitution into three cases according to m and n being odd or even;

Case 1 If ***m* is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\,dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x\,dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\,dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\,dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example#1: Find $\int \sin^3 x \cos^2 x\,dx$ (example on Case 1, where m is odd)

Solution:

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx \qquad m \text{ is odd.}$$
$$= \int (1-\cos^2 x)\cos^2 x\,(-d(\cos x)) \qquad \sin x\,dx = -d(\cos x)$$
$$= \int (1-u^2)(u^2)(-du) \qquad u = \cos x$$
$$= \int (u^4 - u^2)\,du \qquad \text{Multiply terms.}$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

Example#2: Evaluate $\int cos^5 x\, dx$

Solution: This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad \cos x\, dx = d(\sin x)$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

Example#3: Evaluate $\int sin^2 x\, cos^4 x\, dx$

Solution: This is an example of Case 3

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x)\cos 2x\, dx \qquad u = \sin 2x,\; du = 2\cos 2x\, dx$$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omitting } C$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Example#4: Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

Solution: To eliminate the square root, we use the identity;

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \text{or} \qquad 1+\cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}.$$

Example#5: Evaluate $\int \tan^4 x\, dx$

Solution:

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

Note that;

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

Example#6: Evaluate $\int \sin 3x \cos 5x \; dx$

Solution: With, $m = 3$ and $n = 5$, we get;

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

H.W.: Evaluate the following integrals;

1) $\int_{-\pi}^{\pi} \sin 3x \sin 3x \; dx$ Answer: π

2) $\int sin^2\theta \cos 3\theta \; d\theta$ Answer: $\frac{1}{6}\sin 3\theta - \frac{1}{4}\sin\theta - \frac{1}{20}\sin 5\theta + C$

3) $\int \frac{sex^3 x}{\tan x}\; dx$ Answer: $\sec x - \ln|\csc x + \cot x| + C$

---

4) *Trigonometric Substitutions*

This method occurs when we replace the variable of integration by a trigonometric functions; $x = a\, tan\theta, x = a\, sin\theta, and\ x = a\, sec\theta$, which are used for transforming integrals like; $\sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, and\ \sqrt{x^2 - a^2}$ into simple integrals, Now,

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

Example#1: Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

Solution: we assume, $x = 2\,tan\theta, \quad dx = 2\,sec^2\theta\, d\theta$

So, $4 + x^2 = 4 + 4\,tan^2\theta = 4(1 + tan^2\theta) = 4\,sec^2\theta$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

Example#2: Evaluate $\int \frac{x^2dx}{\sqrt{9-x^2}}$

Solution: we assume $x = 3\,sin\theta, \quad dx = 3\,cos\theta\, d\theta$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

Example#3: Evaluate $\int \frac{dx}{\sqrt{25x^2-4}}$

Solution: we first rewrite the square root as;

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)}$$
$$= 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$

To put under the square root in the form of x^2-a^2;

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta,$$

$$x^2-\left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$
$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2-\left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$
$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta+\tan\theta| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right| + C.$$

H.W.: Evaluate the integrals;

1) $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$ Answer: $\pi/6$

2) $\int \frac{dx}{\sqrt{4x^2-49}}$ Answer: $\frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right| + C$

5) Integration of Rational Function by Partial Fractions

Here we show how to express a "**rational function**" like $\frac{5x-3}{x^2-2x-3}$, which is difficult to integrate, as a sum of simpler form, called "**partial fraction**" like $\frac{2}{x+1}+\frac{3}{x-3}$, which is easy to integrate. So, $\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1}+\frac{3}{x-3}$

Then, we can integrate;

$$\int \frac{5x-3}{(x+1)(x-3)}dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$$
$$= 2\ln|x+1| + 3\ln|x-3| + C$$

Benefitting from analyzing (x^2 - 2x - 3) into (x + 1) * (x - 3) using any common method, therefore;

$$\frac{5x-3}{x^2-2x-3}=\frac{A}{x+1}+\frac{B}{x-3}$$

To write the equation in this form multiplying both sides by ($x^2 - 2x - 3$), we get;

$$5x-3=A(x-3)+B(x+1)=(A+B)x-3A+B$$

$$A+B=5, \qquad -3A+B=-3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$

Then integrate the new two-part simple function to get the result.

Example#1: Use partial fraction to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx$

Solution:

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned}x^2+4x+1&=A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)\\&=A(x^2+4x+3)+B(x^2+2x-3)+C(x^2-1)\\&=(A+B+C)x^2+(4A+2B)x+(3A-3B-C).\end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned}&\text{Coefficient of } x^2: && A+B+C=1\\&\text{Coefficient of } x^1: && 4A+2B=4\\&\text{Coefficient of } x^0: && 3A-3B-C=1\end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx=\int\left[\frac{3}{4}\frac{1}{x-1}+\frac{1}{2}\frac{1}{x+1}-\frac{1}{4}\frac{1}{x+3}\right]dx$$

$$=\frac{3}{4}\ln|x-1|+\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln|x+3|+K,$$

Example#2: Use partial fraction to evaluate $\int \frac{6x+7}{(x+2)^2}dx$

Solution:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$A = 6$ and $2A + B = 12 + B = 7,$ or $\boxed{A = 6}$ and $\boxed{B = -5}$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right)dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C.$$

Example#3: Use partial fraction to evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

Solution: Note that the numerator has higher power in x than the denominator.

First we divide the numerator into the denominator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \qquad\qquad\qquad \\ x^2-2x-3\overline{)\,2x^3-4x^2-x-3} \\ 2x^3-4x^2-6x \\ \hline 5x-3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx = \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

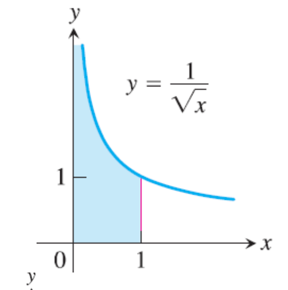
H.W.: Use partial fraction method to evaluate the following integrals;

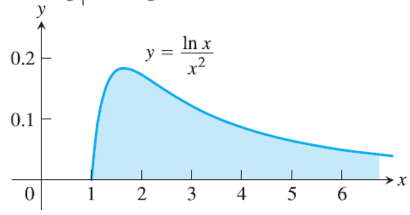
1) $\int \frac{5x-13}{(x-3)(x-2)}dx$ Answer: $\frac{2}{x-3}+\frac{3}{x-2}$

2) $\int \frac{x+4}{(x+1)^2}dx$ Answer: $\frac{1}{x+1}+\frac{3}{(x+1)^2}$

3) $\int \frac{t^2+8}{t^2-5t+6}dt$ Answer: $1+\frac{17}{t-3}+\frac{-12}{t-2}$

6) Improper Integrals

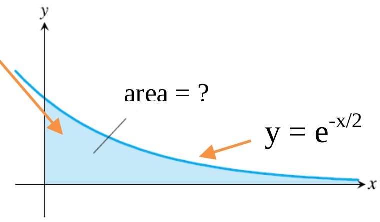




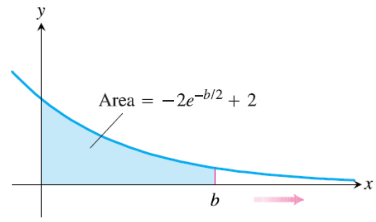
Consider the infinite region that lies under the curves $y = \frac{1}{\sqrt{x}}$ for the range $0 \to 1$ and $y = \frac{\ln x}{x^2}$ for the range $1 \to \infty$ in the first quadrant. You might think that these regions have infinite areas, but we will see that the values are finite.

To solve this problem, for example consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant. First find the area $A(b)$ of the portion from $x = 0$ to $x = b$,

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$



(a)



(b)

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} \left(-2e^{-b/2} + 2\right) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

Example#1: Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution:

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

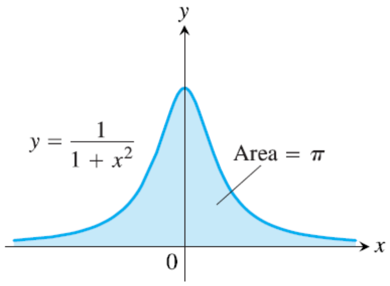
$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

l'Hôpital's Rule

Thus, the improper integral converges and the area has finite value 1.

Example#2: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$



Solution:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0$$

$$= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

H.W.: Evaluate: a) $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\, d\theta$ Answer: $\sqrt{3}$

b) $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ Answer: π

APPLICATIONS FOR INTEGRATION

1- Motion Problems

Example#1: A heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time t during its motion was given as $v(t) = 160 - 32t$ ft/sec.

(a) Find the displacement of the rock during the time period $0 \leq t \leq 8$ sec.

(b) Find the total distance traveled during this time period.

Solution: (a) The displacement can be calculated from the definition of velocity:

$$v = \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \text{ OR } ds = v\,dt, \text{ integrating both sides, gives;}$$

$$\int ds = \int v\,dt, \text{ OR } s = \int v\,dt, \text{ so; } \int_0^8 v(t)\,dt = \int_0^8 (160 - 32t)\,dt = \left[160t - 16t^2\right]_0^8$$

$$= (160)(8) - (16)(64) = 256.$$

This means that the height of the rock is 256 ft. above the ground 8 sec after the explosion.

(b) To find the time required to stop the rock in upward direction and begins to fall downward, we assume $v(t) = 0 = 160 - 32t$, which gives $t = 160/32 = 5$ *sec*, and the displacement in downward direction gets minus sign. So the distance represented by adding the two parts of displacement;

$$\int_0^8 |v(t)|\,dt = \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt$$

$$= \int_0^5 (160 - 32t)\,dt + \left|\int_5^8 (160 - 32t)\,dt\right|$$

$$= \left[160t - 16t^2\right]_0^5 + \left|\left[160t - 16t^2\right]_5^8\right|$$

$$= [(160)(5) - (16)(25)] + \left|[(160)(8) - (16)(64) - ((160)(5) - (16)(25))]\right|$$

$$= 400 + |(-144)| = 544.$$

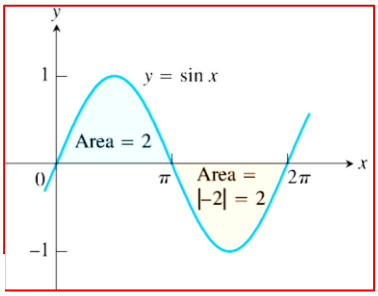
The total distance of **544** ft travelled by the rock during the time period $0 \leq t \leq 8$ sec. is (i) the maximum height of **400** ft is reached over the time interval [0, 5] plus (ii) the additional distance of **144** ft the rock fell over the time interval [5, 8].

2- Area under Curve

Example#1: The figure below shows the graph of the function $f(x) = sinx$ between $x = 0$ and $x = 2\pi$. Compute;

(a) The definite integral of $f(x)$over $[0, 2\pi]$,

(b) The area between the graph of *f(x)* and the *x*-axis over $[0, 2\pi]$.



Solution: The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\, dx = -\cos x \Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x\, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example#2: Find the area of the region between the *x*-axis and the graph of *f(x)*, where,

$$f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2$$

Solution: First find the zeros of f. Since

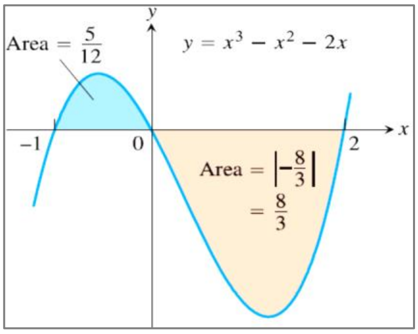
$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0$,-1, and 2. The zeros subdivide the interval [-1,2] into two subintervals: [-1,0], on which $f \ge 0$, and [0,2], on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

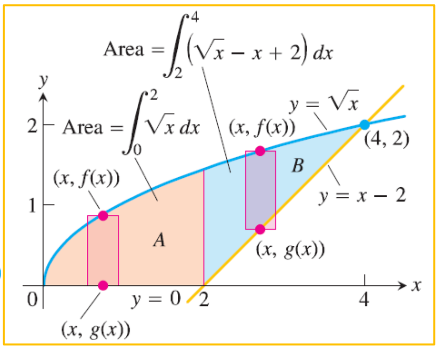
$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$



H.W.: The temperature T (°F) of a room at time t minutes is given by:
$T = 85 - 3\sqrt{25 - t}$ for $0 \le t \le 25$. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$. (Note that temperature between t = 0, 16, and 25 represents area under the curve)

Answer: $T(0) = 70°F$, $T(16) = 76°F$, $T(25) = 85°F$

Example#3: Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



Solution: We can subdivide the region at $x = 2$ into subregions A and B, as shown.

To find the intersection point for the curve and the line;

$\sqrt{x} = x - 2$	Equate $f(x)$ and $g(x)$
$x = (x - 2)^2 = x^2 - 4x + 4$	Square both sides.
$x^2 - 5x + 4 = 0$	Rewrite.
$(x - 1)(x - 4) = 0$	Factor.
$x = 1, \quad x = 4.$	Solve.

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the total area:

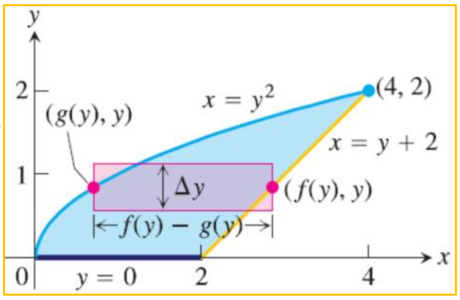
$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Example#4: Find the area of the region in *Ex#3* by integrating with respect to *y*.



Solution: To find the upper limit of *y*;

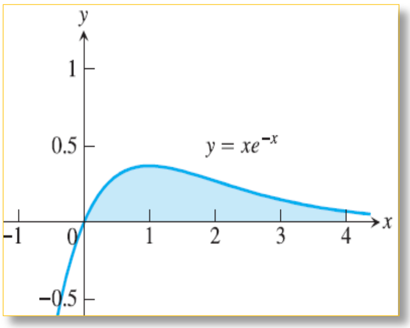
$$y + 2 = y^2 \qquad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$
$$y^2 - y - 2 = 0 \qquad \text{Rewrite.}$$
$$(y + 1)(y - 2) = 0 \qquad \text{Factor.}$$
$$y = -1, \quad y = 2 \qquad \text{Solve.}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the *x*-axis.)

The area of the region is

$$A = \int_c^d [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy$$
$$= \int_0^2 [2 + y - y^2]\,dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$

Example#5: Find the area of the region bounded by the curve $y = xe^{-x}$ and the *x*-axis from $x = 0$ to $x = 4$.



Solution: Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$
$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} - e^{-x}\Big]_0^4$$
$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.$$

H.W.#1: Find the areas of the regions enclosed by the curve and the line given by;

$y = sinx, \quad y = x, \quad 0 \le x \le \pi/4$

Answer: $\frac{\pi^2}{32} + \frac{\sqrt{2}}{2} - 1$

H.W.#2: Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above $y = 2$.

Answer: $\frac{8\sqrt{2} - 7}{6}$

3- Length of a Curve $y = f(x)$

DEFINITION If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

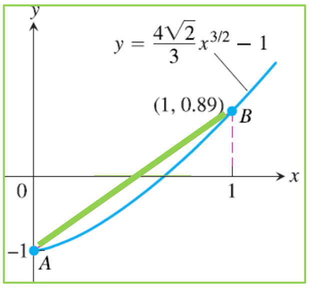
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example#1: Find the length of the curve; $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$

Solution:

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$x = 1, y \approx 0.89$



$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

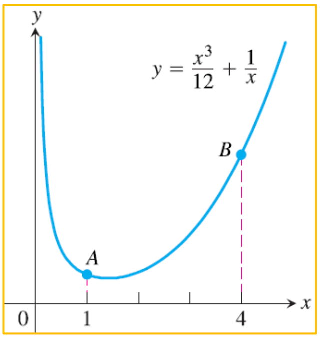
$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17.$$

$a = 0, b = 1$
Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

Example#2: Find the length of the curve; $f(x) = \frac{x^3}{4} - \frac{1}{x^2}, \quad 1 \le x \le 4$

Solution: drawing the graph will simplify the problem,



$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

The length of the graph over $[1, 4]$ is

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6.$$

H.W.: Find the length of the following curve;

$$y = \left(\frac{3}{4}\right)x^{\frac{4}{3}} - \left(\frac{3}{8}\right)x^{\frac{2}{3}} + 5, \quad 1 \le x \le 8$$

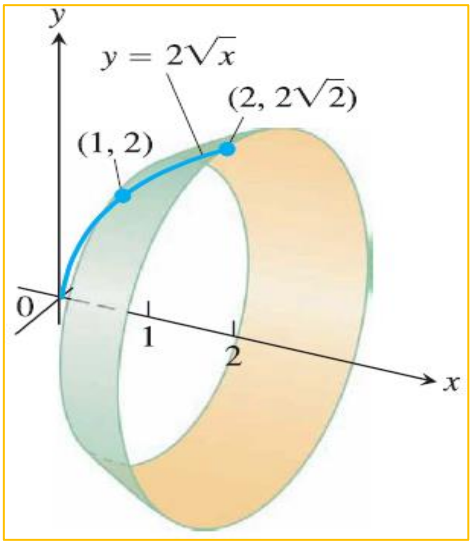
Answer: $\frac{99}{8}$

4- Area of Surface of Revolution

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Example#1: Find the area of the surface generated by revolving the curve; $y = 2\sqrt{x}, \quad 1 \leq x \leq 4$, about the x-axis.



Solution: drawing the graph will simplify the problem,

We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where, $a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

With these substitutions, we have

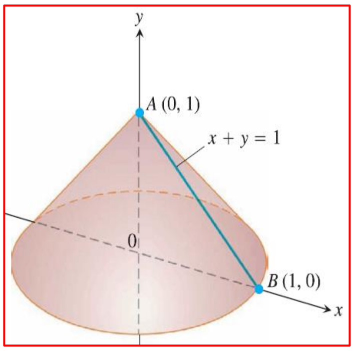
$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi\int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

Surface Area for Revolution About the y-Axis
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy.$$

Example#2: The line segment $x + y = 1, \quad 1 \le y \le 1$, is revolved about the y-axis to generate the cone shown. Find its lateral surface area (which excludes the base area)



Solution: $c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2}.$$

Also, there is a general law for calculating the cone's lateral surface area:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

slant = slope

H.W#1.: Find the area of the surfaces generated by revolving the curve $x = {y^3}/{3}, 0 \le y \le 1$ about y-axis.

Ans. = $\pi(\sqrt{8} - 1)/9$

H.W.#2: Find the surface area of the cone frustum generated by revolving the line segment $y = \left(\frac{x}{2}\right) + \left(\frac{1}{2}\right), \; 1 \le x \le 3$, about the x-axis. Check your result with the geometry formula: $\boxed{\text{frustum surface area} = \pi(r_1 + r_2) * \text{slant height}}$

Ans. = $3\pi\sqrt{5}$

(GOOD LUCK)